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MATHEMATICS Spectrum

TARGET 2017

Rapid
Concept Revision
Cartesian Coordinate System
Straight Line
Pair of Straight Line
Circle
Parabola
Ellipse

FROM CLASS 11th SYLLABUS

Indefinite Integration
Definite Integration
Application of Integrals
Differential Equation

FROM CLASS 12th SYLLABUS

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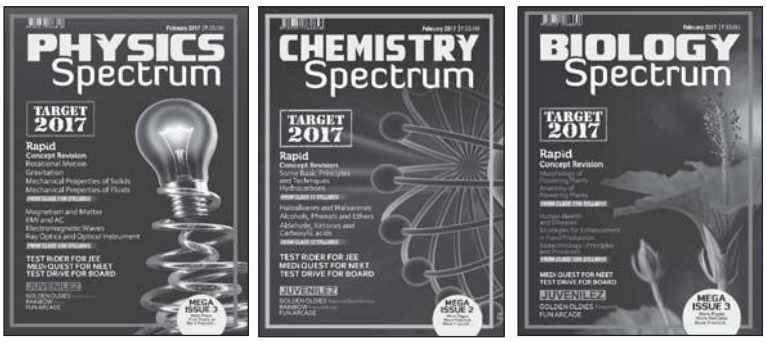

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Issue 3 **Vol 3**

Dear Aspirants,

The month of February is very special as it is the last formal month of preparations. Mathematics is a subject that has many long range objectives. This subject is around us since the beginning of time till today. It lies in nature to all real applications of our daily life from numbering any fruit to how much water we add to cook rice? Or by how many inches you have grown in last few years? Or how much amount is left in bank? All questions can be answered through Mathematics only.

The functioning of Mathematics involves understanding a concept which is followed by practice of its various applications through problems. This subject does not require an ability to memorise, but instead extreme devotion is required towards it. Some general tips to grip this subject are given below:

- Homework and self study both are important.
- Learning by halves is impossible in Maths.
- Question pattern and problems types are predictable in Mathematics.
- Practice is the key.
- Aggressiveness pays a lot.
- Try to minimise the use of hints, calculators etc.
- Challenge yourself and remain enthusiastic.
- Do not hesitate to start again, if you are not sure of procedure/solution.
- Testing helps in filtering wrong and right.
- Strategise or plan up in accordance with your strengths and weakness.

Hope above written tips will prove helpful in targeting your goal and from the side of spectrum family, I personally assure you that we will try to create more and more to help you out. Remember! One more thing

"Life is a test if you woke up alive this morning, it means you got an 'A' for yesterday."
Ron Fry

The spectrum team will be anxiously waiting for your feedbacks and comments.
Good luck

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@ CLASS XI SYLLABUS

Rapid CONCEPT REVISION

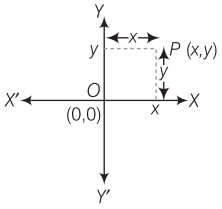
TARGET 2017

CARTESIAN COORDINATE SYSTEM

There are various types of coordinate systems to represent a point in two dimensions, but here we will only discuss cartesian coordinate system and polar coordinate system.

Cartesian Coordinate System

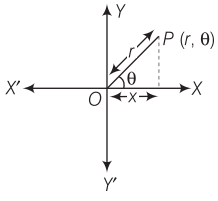
If we draw two mutually perpendicular lines in a plane, then they divide the plane in four parts, each part is known as I, II, III and IV quadrants and their point of intersection is called origin. Horizontal and vertical lines are known as X and Y-axes, respectively. Every point in the plane is located in the form of ordered pair (x, y), where x denotes the distance of the point from Y-axis and y is distance from X-axis. If a point lies on X-axis, then it is in the form of $(x, 0)$ and on Y-axis, it is $(0, y)$.



Since, $X'OX$ is perpendicular to YOY', so this system of representation is also called rectangular coordinate system or orthogonal coordinate system.

Polar Coordinate System

Another way to locate the point in the plane is polar form (r, θ). Here, r is the distance of the point from the origin and θ is the angle of inclination of radius vector (**OP**) with the positive X-axis. If coordinates of the point P in cartesian and polar form are (x, y) and (r, θ) respectively, then



$$x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\therefore \quad \tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Some Basic Results

(i) **Distance formula** If two points are $A(x_1, y_1)$ and $B(x_2, y_2)$, then $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and if $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ are polar coordinates, then $AB^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$.

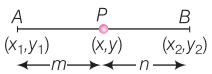
SPEED BOOSTER

- Distance between $(x_1, 0)$ and $(x_2, 0)$ is $|x_1 - x_2|$.
- Distance between $(0, y_1)$ and $(0, y_2)$ is $|y_1 - y_2|$.

(ii) **Section formulae** If two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given and $P(x, y)$ is any point, which divides the line segment AB in the ratio of $m : n$, then we can find the coordinates of $P(x, y)$.

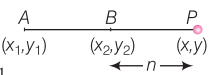
(a) If P lies in between A and B (internally), then

$$x = \frac{mx_2 + nx_1}{m + n} \text{ and } y = \frac{my_2 + ny_1}{m + n}$$



(b) If P lies on AB produced (externally), then

$$x = \frac{mx_2 - nx_1}{m - n} \text{ and } y = \frac{my_2 - ny_1}{m - n}$$

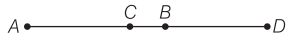


Note If (x_1, y_1) and (x_2, y_2) are the coordinates of points A and B respectively, then the coordinates of mid-point of line segment joining A and B are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

SPEED BOOSTER

- To find ratio, consider P divides the given line segment in $k:1$, then coordinates of P are given by $\left(\frac{x_1 + kx_2}{k+1}, \frac{y_1 + ky_2}{k+1}\right)$. If k is positive, then it divides the line segment internally and for negative, it divides the line segment externally.
- To prove that four points A, B, C and D are vertices of
  (a) **Parallelogram** show that diagonals AC and BD have same mid-point.
  (b) **Rhombus** show that diagonals have same mid-point and adjacent sides are equal.
  (c) **Square** show that diagonals are equal and adjacent sides are equal.
  (d) **Rectangle** show that diagonals are equal and they have same mid-point.

(iii) **Harmonic conjugates** Let A and B be two given points. Now, if the point C divides line segment AB internally in the ratio $m:n$ and the point D divides the line segment AB externally in the ratio $m:n$, then the points C and D are called the harmonic conjugates with respect to points A and B.



Note $|AC|, |AB|, |AD|$ are in harmonic progression,

i.e. $$\frac{2}{|AB|} = \frac{1}{|AC|} + \frac{1}{|AD|}$$

(iv) **Area of triangle** If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of triangle, then the area of triangle is given by

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

or $$\frac{1}{2}\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

Note (i) If three points A, B and C are collinear, then area of ΔABC is zero and *vice-versa*.
(ii) Triangle having integral coordinates can never be equilateral.

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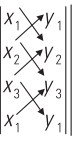
- Area of triangle, having vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, can also be found by easy method, i.e. Stair method which is given below:

Area of $\Delta ABC = \frac{1}{2}$



$$= \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}\right|$$

- If $a_r x + b_r y + c_r = 0\ (r = 1, 2, 3)$ are the equations of sides of a triangle, then the area of the triangle is given by

$$\frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2, C_3 are the cofactors of c_1, c_2, c_3 in the determinant.

- Area of triangle formed by coordinate axes and the line $ax + by + c = 0$ is $\frac{c^2}{2ab}$.

(v) **Area of polygon** If $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ are the vertices of a polygon, then area of polygon is given by

$$\frac{1}{2}\left\|\begin{matrix} x_1 & x_2 \\ y_1 & y_2 \end{matrix}\right| + \left|\begin{matrix} x_2 & x_3 \\ y_2 & y_3 \end{matrix}\right| + \left|\begin{matrix} x_3 & x_4 \\ y_3 & y_4 \end{matrix}\right| + \ldots + \left|\begin{matrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{matrix}\right| + \left|\begin{matrix} x_n & x_1 \\ y_n & y_1 \end{matrix}\right\|$$

or $$\frac{1}{2}\left|\{(x_1y_2 - x_2y_1) + (x_2y_3 - y_2x_3) + (x_3y_4 - y_3x_4) + \ldots + (x_{n-1}y_n - y_{n-1}x_n) + (x_ny_1 - y_nx_1)\}\right|$$

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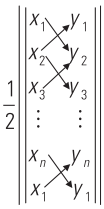
- Using Stair method, we have

Area of polygon $= \frac{1}{2}$

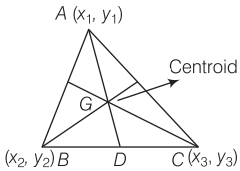


$$= \frac{1}{2}\left|\{(x_1y_2 + x_2y_3 + \ldots + x_ny_1) - (x_2y_1 + x_3y_2 + \ldots + x_1y_n)\}\right|$$

- If the two opposite vertices of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area $= \frac{1}{2}AC^2 = \frac{1}{2}[(x_2 - x_1)^2 + (y_2 - y_1)^2]$
- Area of rhombus formed by $|ax| + |by| + |c| = 0$ is $\frac{2c^2}{ab}$.

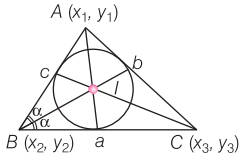
Standard Points of a Triangle

(i) **Centroid** The point of intersection of the medians of triangle is called the centroid of the triangle. If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are vertices of the triangle, then coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.



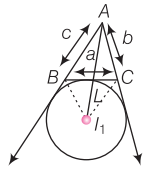
(ii) **Incentre** The point of intersection of internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre of a triangle with vertices as $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$, are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$



where, $a = BC$, $b = AC$ and $c = AB$.

(iii) **Excentre** The point of intersection of external bisector of the two angles and internal angle bisector of third angle is called excentre. Hence, for a triangle, three excentres are possible, opposite to each vertex.



(a) Excentre opposite to A

$$= \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

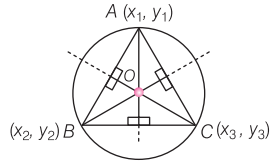
(b) Excentre opposite to B

$$= \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

(c) Excentre opposite to C

$$= \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

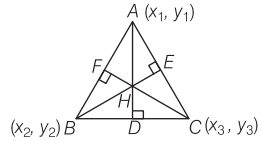
(iv) **Circumcentre** The point of intersection of perpendicular bisectors of the sides of a triangle is called circumcentre of triangle. This is the point which is at equidistant from the vertices of the triangle.



If O is circumcentre, then $AO = BO = CO$ will give two equations. By simplifying it, we can get coordinates of circumcentre. Also, we can get coordinates of circumcentre by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

(v) **Orthocentre** The point of intersection of the altitudes drawn from the vertices of the triangle is called orthocentre of the triangle.



If H is orthocentre of the ΔABC, then coordinates of orthocentre are

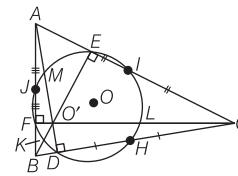
$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

SPEED BOOSTER

- If angle bisectors of ΔABC intersect BC, AC and AB at D, E and F, respectively. Also, $|AB| = c, |BC| = a, |CA| = b$, then
  $AF = AE = s - a, BF = BD = s - b$ and $CD = CE = s - c$
  where, s is semi-perimeter of ΔABC.
- Orthocentre of the triangle having vertices (a, b), (b, a) and (a, a), is (a, a).
- If the circumcentre and centroid of a triangle are respectively (a, b) and (c, d), then orthocentre is $(3c - 2a, 3d - 2b)$.

(vi) **Nine point centre** Let ABC be a triangle such that AD, BE and CF are its altitudes drawn from A, B and C on the opposite sides, H, I, J are the mid-points of the sides BC, CA, AB respectively and K, L, M are the mid-points of the line segments joining the orthocentre O' to the angular points A, B, C. These nine points $(D, E, F, H, I, J, K, L, M)$ are concyclic and the circle passing through these nine points is called the nine point circle. Its centre is known as the nine point centre.

The nine point centre of a triangle is collinear with the circumcentre O and the orthocentre O' and bisects the segment joining them. The radius of nine point circle is half the radius of the circumcircle.



REMEMBER

- Centroid of the triangle formed by joining the middle points of the sides of a triangle is same as the centroid of the triangle.
- Centroid divides the median in 2 : 1 (2 from the vertex and 1 from the opposite side).
- Incentre divides the angle bisector in following ratios
  $b + c : a, c + a : b, a + b : c$
- Circumcentre, centroid and orthocentre of an acute triangle are collinear. Also, centroid divides the join of orthocentre and circumcentre in the ratio 2 : 1.
- In an equilateral triangle, centroid, orthocentre, incentre and circumcentre coincide. In an isosceles triangle, all these four points lie on the same line.
- In right angled triangle, orthocentre lies at the vertex which forms right angle.

Locus

The path described by a point which moves under given geometrical condition(s) is called its locus.

Some important loci are as follow :

(i) Locus of a point, which is equidistant from a fixed point, is circle.

(ii) If sum of the distances of a point from two fixed points is always same, then locus is called ellipse. If A and B are two fixed points and P is any point such that $PA + PB = 2a$, then P will form ellipse.

(iii) If difference of the distances of a point from two fixed points is always same, then locus is called hyperbola. If A and B are two fixed points and P is any point such that $|PA - PB| = 2a$, then P will form hyperbola.

(iv) Locus of point which is equidistant from two fixed points is called perpendicular bisector. If A and B are two fixed points and P is any point such that $PA = PB$, then the locus of P will be perpendicular bisector of line segment AB.

Process to Find the Locus of a Point

Step I Let (h, k) be the coordinates of the moving point, say P.

Step II Apply the geometrical conditions on (h, k) and establish the relation between h and k.

Step III Now, replace h by x and k by y, then the obtained equation is the required locus.

Transformation or Change of Axes

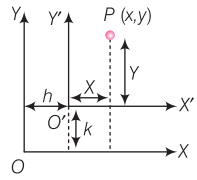
The coordinates of the points and equation of the curve changes when either the origin is changed or direction of axes or both are changed. This process is called transformation of axes or change of axes.

Change of Origin (Translation of Axes)

If (x, y) are the coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then $x = X + h$ and $y = Y + k$, where (h, k) are coordinates of shifted origin.

(i) The point whose coordinates are (x, y), has now the coordinates $(x - h, y - k)$.

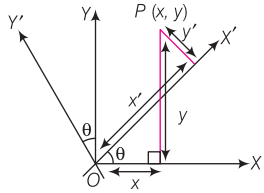
(ii) When origin O is shifted to point $O'(h, k)$ without rotation of axes, then new equation of curve can be obtained by putting $(X + h)$ in place of x and $(Y + k)$ in place of y.



Rotation of Axes without Changing the Origin

Let OX, OY be the original axes and OX', OY' be the new axes obtained by rotating OX and OY through an angle θ in anti-clockwise sense.

Let P be any point in the plane having coordinates (x, y) with respect to OX and OY and (x', y') with respect to axes OX' and OY'.



Then, $x = x' \cos\theta - y' \sin\theta$; $y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$; $y' = -x \sin\theta + y \cos\theta$

Relation between (x, y) and (x', y')

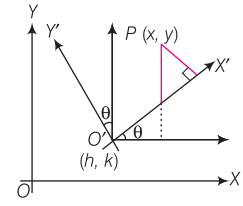
	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Note (i) The axes rotation in anti-clockwise direction is positive and clockwise rotation of axis is negative.

(ii) In order to remove the xy term from $f(x, y) = ax^2 + 2hxy + by^2$ without changing the origin, rotate the axes through an angle θ, where θ is given by $\tan 2\theta = \frac{2h}{a - b}$ or $\cot 2\theta = \frac{a - b}{2h}$ according as $a \neq b$ or $a = b$.

Change of Origin and Rotation of Axes

If origin O is changed to $O'(h, k)$ and axes are rotated about the new origin O' by an angle θ in the anti-clockwise sense such that new coordinates of $P(x, y)$ become (x', y'), then the equations of transformation will be

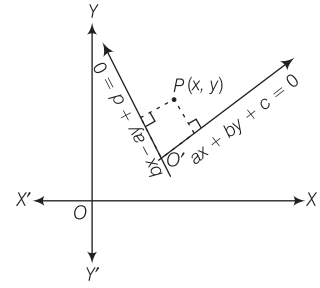


$$x = h + x' \cos\theta - y' \sin\theta$$

and $$y = k + x' \sin\theta + y' \cos\theta$$

REMEMBER

If $P(x, y)$ and two mutually perpendicular lines $ax + by + c = 0$ and $bx - ay + d = 0$ are taken as new axes such that new coordinates of P are (x', y'), then



$$x' = \frac{bx - ay + d}{\sqrt{b^2 + a^2}}$$

and $$y' = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

MASTER STROKES

1. The coordinates of the point where origin should be shifted so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term, are

(a) $\left(\frac{3}{4}, -2\right)$ (b) $\left(-\frac{3}{4}, 2\right)$

(c) $\left(2, -\frac{3}{4}\right)$ (d) $\left(-2, \frac{3}{4}\right)$

2. If the axes is turned through an angle $\tan^{-1} 2$, then the equation $4xy - 3x^2 = a^2$ transforms to

(a) $X^2 + 4Y^2 = a^2$ (b) $X^2 - 4Y^2 = 4a^2$

(c) $X^2 - 4Y^2 = a^2$ (d) $X^2 + 4Y^2 = 4a^2$

3. If two vertices of an equilateral triangle have integral coordinates, then the third vertex will have

(a) integral coordinates

(b) coordinates which are rational

(c) atleast one coordinate which is irrational

(d) coordinates which are irrational

4. If points $(a^2, 0)$, $(0, b^2)$ and $(1, 1)$ are collinear, then

(a) $\frac{1}{a^2} + \frac{1}{b^2} = 1$ (b) $\frac{1}{a} + \frac{1}{b} = 1$

(c) $a^2 + b^2 = 1$ (d) None of these

5. If the area of a triangle formed by the points $O(0, 0)$, $A(a^{x^2}, 0)$ and $B(0, a^{6x})$ is $\frac{1}{2a^5}$ sq unit, then x equals

(a) 1, 5 (b) – 1, 5

(c) 1, – 5 (d) – 1, – 5

6. The area of the triangle with vertices at $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$, is

(a) 0 (b) $(a + b + c)$ sq units

(c) $(ab + bc + ca)$ sq units (d) None of these

7. If the point (x, y) is equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$, then

(a) $ax = by$ (b) $bx = ay$

(c) $ax + by = 0$ (d) $bx + ay = 0$

8. If P and Q are points whose coordinates are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ respectively and S is the point $(a, 0)$. Then, harmonic mean of SP and SQ is

(a) a (b) $4a$

(c) $2a$ (d) $2 / a$

9. If the coordinates of the mid-points of sides AB and AC of ΔABC are $D(3, 5)$ and $E(-3, -3)$ respectively, then BC equals

(a) 10 (b) 15

(c) 20 (d) 30

10. If O is the origin and $P(x_1, y_1)$, $Q(x_2, y_2)$ are two points, then $OP \times OQ \sin \angle POQ$ equals

(a) $x_1x_2 + y_1y_2$ (b) $x_1y_2 + x_2y_1$

(c) $|x_1y_2 - x_2y_1|$ (d) None of these

11. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$, is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

12. The x-coordinate of the incentre of a triangle, where the mid-points of the sides are $(0, 1)$, $(1, 1)$ and $(1, 0)$, is

(a) $2 + \sqrt{2}$ (b) $1 + \sqrt{2}$

(c) $2 - \sqrt{2}$ (d) $1 - \sqrt{2}$

13. If $OPQR$ is a square and M, N are the mid-points of the sides PQ and QR respectively, then the ratio of the areas of the square and the ΔOMN, is

(a) 4 : 1 (b) 2 : 1

(c) 8 : 3 (d) 4 : 3

14. Let $O(0, 0)$, $P(3, 4)$ and $Q(6, 0)$ be the vertices of ΔOPQ. The point R inside the ΔOPQ is such that ΔOPR, ΔPQR and ΔOQR are of equal area. Then, the coordinates of R are

(a) (4/3, 3) (b) (3, 2 / 3)

(c) (3, 4/3) (d) (4/3, 2/3)

15. Consider three points $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \theta, \alpha, \beta < \frac{\pi}{4}$.

Then,

(a) P lies on the line segment RQ

(b) Q lies on the line segment PR

(c) R lies on the line segment QP

(d) P, Q, R are non-collinear

16. Let $A(5, 12)$, $B(-13\cos\theta, 13\sin\theta)$ and $C(13\sin\theta, -13\cos\theta)$ be the vertices of ΔABC, where $\theta \in R$. Then, the locus of the orthocentre of ΔABC is

(a) $x - y + 7 = 0$ (b) $x - y - 7 = 0$

(c) $x + y - 7 = 0$ (d) $x + y + 7 = 0$

Answers

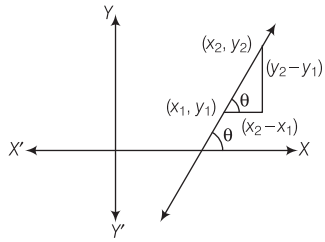
1. (a)	2. (c)	3. (c)	4. (a)	5. (d)
6. (a)	7. (b)	8. (c)	9. (c)	10. (c)
11. (d)	12. (c)	13. (c)	14. (c)	15. (d)
16. (a)				

STRAIGHT LINE

A straight line can be defined as the locus of a point satisfying the condition $ax + by + c = 0$, where a, b, c are constants.

Slope (Gradient) of a Line

In Mathematics, the slope or gradient of a line is a number which is used to indicate the steepness of a line as well as indicating whether the line is tilled uphill or downhill. Slope of a line is denoted by 'm'.



If a line passing through (x_1, y_1) and (x_2, y_2) and makes an angle (say θ) with positive direction of X-axis in anti-clockwise direction, then slope of a line is $m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \tan\theta$.

REMEMBER

- Slope of the line parallel to X-axis is $m = \tan 0° = 0$ and slope of the line parallel to Y-axis is $m = \tan 90° = \infty$ (not defined).
- Slope of a line equally inclined with axes is 1 or -1 as it makes an angle of 45° or 135° respectively, with X-axis.
- Slope of the line $ax + by + c = 0$ is $-a/b$.
- If m_1 and m_2 are the slopes of two lines , then
  (a) lines are parallel iff $m_1 = m_2$.
  (b) lines are perpendicular iff $m_1 m_2 = -1$.

Various Form of Equation of Straight Line

A line is determined uniquely by anyone of the following ways:

(i) If two different points are known, then we can draw a unique straight line.

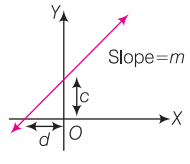
(ii) If a point and its inclination or slope is given.

Let us look at the different forms of equation of straight line.

(i) **Slope form** Equation of line through the origin and having slope m is $y = mx$.

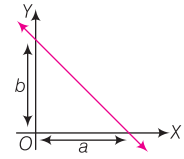
(ii) **Point and slope form** Equation of a line through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

(iii) **Slope intercept form** Equation of line with slope m and makes an intercept c on Y-axis is $y = mx + c$.
If a line makes intercept d with X-axis, then $y = m(x - d)$.

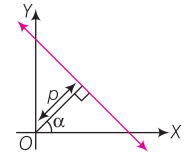


(iv) **Two points form** If (x_1, y_1) and (x_2, y_2) are the coordinates of two points, then equation of straight line is $(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

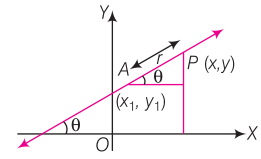
(v) **Intercept form** If a line makes intercept a and b with the coordinate axes, then equation of straight line is $\frac{x}{a} + \frac{y}{b} = 1$.



(vi) **Normal form** If p is length of perpendicular drawn from origin to straight line and this perpendicular makes angle α with positive side of X-axis, then equation of line is $x\cos\alpha + y\sin\alpha = p$.



(vii) **Parametric form** Equation of straight line passing through (x_1, y_1) and making an angle θ with positive direction of X-axis, is $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = \pm r$, where r is distance of the point (x, y) from (x_1, y_1). This form is also known as symmetrical or distance form.



Note The coordinates of any point on this line may be taken as $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.

Point and its Inclination with the Line is Given

The equation of straight line which passes through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$, is $y - y_1 = \left(\frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}\right)(x - x_1)$.

SPEED BOOSTER

- Equation of line, which is parallel to the line $ax + by + c = 0$, is $ax + by + k = 0$.
- Equation of line which is perpendicular to line $ax + by + c = 0$, is $bx - ay + k = 0$.
  We can find value of k by additional information given in question.

General Equation of Straight Line and its Transformation

General equation of straight line is $ax + by + c = 0$.
General equation of straight line can be written in different forms as follow:

(i) **Slope intercept form** $y = -(a/b)x - c/b$, slope $(m) = -a/b$ and intercept of Y-axis is $-c/b$.

(ii) **Intercept form** $\frac{x}{-c/a} + \frac{y}{-c/b} = 1$,
x-intercept is $-c/a$ and y-intercept is $-c/b$.

(iii) **Normal form** To change the general form of a line into normal form, first take c to right side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$ like $\frac{-ax}{\sqrt{a^2+b^2}}+\frac{-by}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}$

Then, $\cos\alpha=\frac{-a}{\sqrt{a^2+b^2}}, \sin\alpha=\frac{-b}{\sqrt{a^2+b^2}}$

and $p=\frac{c}{\sqrt{a^2+b^2}}$

Equation of Family of Lines

Equation of family of lines, passing through the intersection of $L_1\equiv a_1x+b_1y+c_1=0$ and $L_2\equiv a_2x+b_2y+c_2=0$ is given by $L_1+\lambda L_2=0$, where λ is any parameter.

Note Point of intersection of two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is given by

$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-b_1a_2}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)=\left(\frac{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}, \frac{\begin{vmatrix}c_1 & a_1\\ c_2 & a_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}\right)$$

Angle between Two Lines

If $y=m_1x+c_1$ and $y=m_2x+c_2$ are two lines and θ is angle between them, then $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

SPEED BOOSTER

Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ be two given lines.

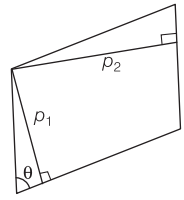
- If θ is the angle between two given lines, then $\tan\theta=\left|\frac{b_1a_2-a_1b_2}{a_1a_2+b_1b_2}\right|$
- Two lines are coincident iff $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$
- Two lines are parallel iff $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$
- Two lines intersect each other at one point iff $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$
- Two lines are perpendicular iff $a_1a_2+b_1b_2=0$

Length of Perpendicular from a Point on a Line

The length of perpendicular from point $P(x_1, y_1)$ to the line $ax+by+c=0$ is $d=\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|_{(x_1, y_1)}=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

SPEED BOOSTER

- If two lines are parallel such as $ax+by+c_1=0$ and $ax+by+c_2=0$, then distance between the lines, $d=\left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$
- The area of parallelogram $=\frac{p_1p_2}{\sin\theta}$, where p_1 and p_2 are distances between two pairs of opposite sides and θ is the angle between any two adjacent sides.



- If $y=m_1x+c_1, y=m_1x+c_2$ and $y=m_2x+d_1, y=m_2x+d_2$, then area of parallelogram formed by these lines $=\left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right|$
- Length of perpendicular from the origin to the line $ax+by+c=0$ is $\left|\frac{a\cdot0+b\cdot0+c}{\sqrt{a^2+b^2}}\right|=\frac{|c|}{\sqrt{a^2+b^2}}$.
- The length of perpendicular from (x_1, y_1) to the line $x\cos\alpha+y\sin\alpha=p$ is $|x_1\cos\alpha+y_1\sin\alpha-p|$.

Position of Points

If ax_1+by_1+c is of the same sign as c, then the point (x_1, y_1) lies on the origin side of $ax+by+c=0$, but if the sign of ax_1+by_1+c is opposite to that of c, then the point (x_1, y_1) lies on the non-origin side of $ax+by+c=0$.

Note Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same side of the line $ax+by+c=0$, if $\frac{ax_1+by_1+c}{ax_2+by_2+c}>0$ and if they lie on opposite side of the line $\frac{ax_1+by_1+c}{ax_2+by_2+c}<0$.

Equation of Bisectors

Equation of the bisector of angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

Acute and Obtuse Angle Bisector

(i) If $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where $c_1, c_2>0$ and $a_1b_2\neq a_2b_1$, then there are following two cases :

***Case* I** If $a_1a_2+b_1b_2>0$, then negative sign gives acute angle bisector and positive sign gives obtuse angle bisector.

***Case* II** If $a_1a_2+b_1b_2<0$, then negative sign gives obtuse angle bisector and positive sign gives acute angle bisector.

(ii) Algorithm to determine the bisector of the angle containing the origin and that of the angle not containing the origin.

***Step* I** Rewrite equations such that constant term are positive.

***Step* II** Positive (+) sign gives angle bisector, which contains the origin and negative (–) sign gives the equation of angle bisector, which does not contain the origin.

i.e. $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

where, $c_1, c_2>0$ gives equation which contains origin and equation

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}},$$

does not contain origin.

(iii) **Algorithm to determine the bisector of the angle containing the given point**
If (h, k) is any point, then the equation of angle bisector which contains a given point, we follow following steps:

***Step* I** Find the value of the equations at given point.

***Step* II** If both are of the same sign, then the equation is

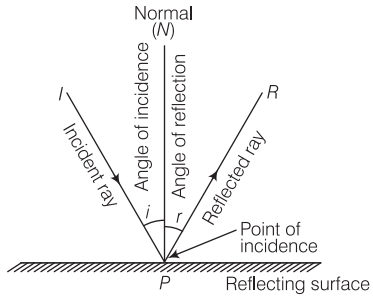
$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

and it contains the point (h, k).
If value of the equations evaluated in Step I is of opposite sign, then the equation is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Reflection of Light

When a ray of light falls on a smooth polished surface (mirror) separating two media, a part of it is reflected back into the first medium.



IP is the incident ray and PR is the reflected ray. A perpendicular drawn to the surface, at the point of incidence P is called the normal. Hence, PN is the normal. The angle between the incident ray and the normal ($\angle IPN$) is called the angle of incidence which is represented by $\angle i$ i.e. $\angle IPN = \angle i$ = Angle of incidence and the angle between the reflected ray and the normal ($\angle RPN$) is called the angle of reflection which is represented by $\angle r$ i.e. $\angle RPN = \angle r$ = Angle of reflection.

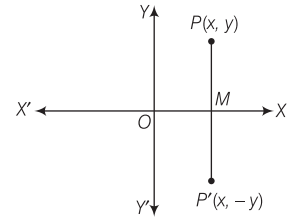
Laws of Reflection

(i) The incident ray, normal and the reflected ray to a surface at the point of incidence all lie in the same plane.
(ii) Angle of incidence = Angle of reflection
i.e. $\angle i = \angle r$

Reflection of a Point under Different Conditions

(i) Reflection of a Point with Respect to *X*-axis

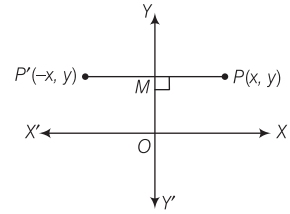
If $P(x, y)$ is any point, then its reflection about X-axis (i.e. in line $y = 0$) will be $P'(x, -y)$.



Note The reflection of the line $ax + by + c = 0$ about X-axis is $ax - by + c = 0$.

(ii) Reflection of a Point with Respect to *Y*-axis

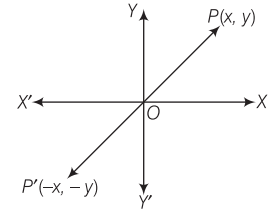
If $P(x, y)$ is any point, then its reflection about Y-axis (i.e. in line $x = 0$) will be $P'(-x, y)$.



Note The reflection of the line $ax + by + c = 0$ about Y-axis is $-ax + by + c = 0$.

(iii) Reflection of a Point with Respect to Origin

If $P(x, y)$ is any point, then its reflection about the origin will be $P'(-x, -y)$.

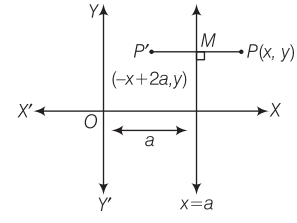


Note The reflection of the line $ax + by + c = 0$ about the origin is $-ax - by + c = 0$.

(iv) Reflection of a Point with Respect to the Line $x = a$

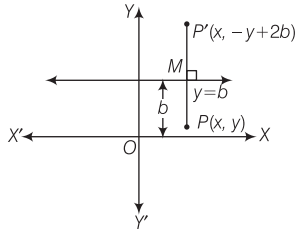
If $P(x, y)$ is any point, then its reflection about the line $x = a$, will be $P'(-x + 2a, y)$.

Note The reflection of the line $ax + by + c = 0$ about the line $x = \lambda$ is $a(2\lambda - x) + by + c = 0$.



(v) Reflection of a Point with Respect to the Line $y = b$

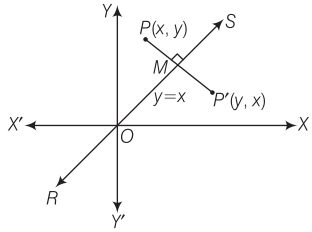
If $P(x, y)$ is any point, then its reflection about the line $y = b$, will be $P'(x, -y + 2b)$.



Note The reflection of the line $ax + by + c = 0$ about the line $y = \mu$ is $ax + b(2\mu - y) + c = 0$.

(vi) Reflection of a Point with Respect to the Line $y = x$

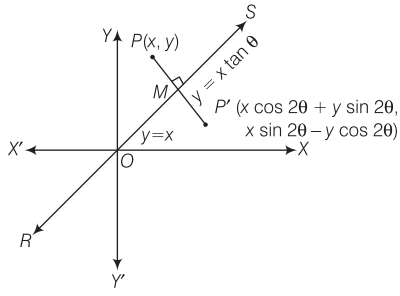
If $P(x, y)$ is any point, then its reflection about the line $y = x$, is $P'(y, x)$.



Note The reflection of the line $ax + by + c = 0$ about the line $y = x$ is $ay + bx + c = 0$.

(vii) Reflection of a Point with Respect to the Line $y = x \tan\theta$

If $P(x, y)$ is any point, then its reflection about the line $y = x \tan\theta$, is $P'(x\cos 2\theta + y \sin 2\theta, x \sin 2\theta - y \cos 2\theta)$.



Note

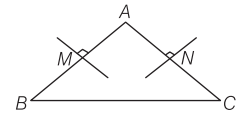
(i) If (x_1, y_1) is any point and $ax + by + c = 0$ is equation of line, then to find the coordinates of foot of perpendicular, use the result $\frac{(x_2 - x_1)}{a} = \frac{(y_2 - y_1)}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$. By simplifying the above result, we can find the values of x_2 and y_2.

(ii) Image of reflection of point (x_1, y_1) about a line mirror $ax + by + c = 0$ is $\frac{(x_2 - x_1)}{a} = \frac{(y_2 - y_1)}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$.

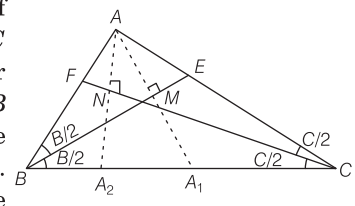
Use of Reflection

To make the following types of problems easier, we use the reflection.

***Type* I** If vertex of a ΔABC and equations of perpendicular bisectors of AB and AC are given, then B and C are the reflections of A about the perpendicular bisectors of AB and AC (where, M and N are the mid-points of AB and AC).

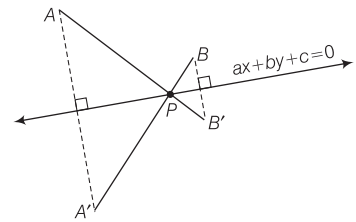


***Type* II** The reflection of vertex A of a ΔABC about the angular bisector of angles B and C lie on the side BC (by congruence). A_1 and A_2 are the images of A about the angle bisectors BE and CF respectively, where M and N are the mid-points of AA_1 and AA_2.



Optimisation
(Minimisation or Maximisation)

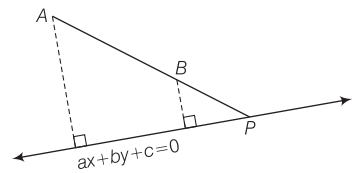
(i) **Minimisation** Let A and B be two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $PA + PB$ is minimum. Then, find the image of A or B about the line $ax + by + c = 0$ (say A' or B'), then join B' with A or A' with B and wherever it intersects $ax + by + c = 0$ is the required point.



$\therefore \quad PA + PB = PA + PB'$ or $PA + PB = PA' + PB$

and $|PA + PB| = |PA + PB'| = |AB'|$ [minimum value]

(ii) **Maximisation** Let A and B be two given points on the same side of $ax + by + c = 0$. Suppose we want to determine a point P on $ax + by + c = 0$ such that $|PA - PB|$ is maximum, then find the equation of line joining A and B and the point at which it intersects $ax + by + c = 0$ is the required point.



and $|PA - PB| = |AB|$ [maximum value]

MASTER STROKES

1. The equation of the line AB is $y = x$. If A and B lie on the same side of the line mirror $2x - y = 1$, then the equation of the image of AB is

(a) $x + y - 2 = 0$ (b) $8x + y - 9 = 0$

(c) $7x - y - 6 = 0$ (d) None of these

2. Let A be the fixed point $(0, 4)$ and $B(2t, 0)$ be a moving point. Let M be the mid-point of AB and the perpendicular bisector of AB meets the Y-axis at R. The locus of the mid-point P of MR is

(a) $x^2 + y^2 = \frac{1}{4}$ (b) $(y - 2)^2 - x^2 = 4$

(c) $y + x^2 = 2$ (d) $3x^2 + y^2 = 8$

3. Let A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively. We define the distance between A and B as $d(A, B) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$. If $d(A, O) = 1$, where O is the origin, then the locus of A has an area of

(a) 1 sq unit (b) 2 sq units

(c) 4 sq units (d) 1/4 sq unit

4. The locus of the centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and $(1, 0)$, where t is a parameter, is

(a) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$

(b) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$

(c) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$

(d) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

5. The equation of the bisector of that angle between the lines $x + y = 3$ and $2x - y = 2$ which contains the point $(1, 1)$, is

(a) $(\sqrt{5} - 2\sqrt{2})x + (\sqrt{5} + \sqrt{2})y - 3\sqrt{5} + 2\sqrt{2} = 0$

(b) $(\sqrt{5} + 2\sqrt{2})x + (\sqrt{5} - \sqrt{2})y - 3\sqrt{5} - 2\sqrt{2} = 0$

(c) $3x = 10$

(d) None of the above

6. The number of integral points (integral point means both the coordinates should be integer) exactly in the interior of the triangle with vertices $(0, 0)$, $(0, 21)$ and $(21, 0)$, is

(a) 133 (b) 190

(c) 233 (d) 105

7. If $A(2, 3)$ and $B(-2, 1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is

(a) $2x + 3y = 1$ (b) $2x + y = 3$

(c) $2x - 3y = 1$ (d) $x - y = 1$

8. The equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is -1, is

(a) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

(b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(c) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{2} + \frac{y}{1} = 1$

(d) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

9. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point $(13, 32)$. If the line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$, then the distance between L and K is

(a) $17/\sqrt{15}$ (b) $23/\sqrt{17}$

(c) $23/\sqrt{15}$ (d) $\sqrt{17}$

10. A straight line L through the point $(3, -2)$ is inclined at an angle of $60°$ to the line $\sqrt{3}x + y = 1$. If L also intersects the X-axis, then the equation of L is

(a) $\sqrt{3}x + y + 2 - 3\sqrt{3} = 0$

(b) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(c) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$

(d) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

11. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then, the set of all possible values of a is in the interval

(a) $[1, \infty)$ (b) $(-1, \infty)$

(c) $(-1, 1)$ (d) $(0, \infty)$

12. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is a/an

(a) hyperbola (b) parabola

(c) ellipse (d) straight line

13. If a ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching X-axis, then the equation of the reflected ray is

(a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$

(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$

14. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(a) $4x + 7y + 3 = 0$ (b) $2x - 9y - 11 = 0$

(c) $4x - 7y - 11 = 0$ (d) $2x + 9y + 7 = 0$

15. If the point $(p, 5)$ lies on the line parallel to Y-axis and passing through the intersection of the lines $2(a^2 + 1)x + by + 4(a^3 + a) = 0$ and $(a^2 + 1)x - 6by + 2(a^3 + a) = 0$, then p is equal to

(a) $3a$ (b) $-2a$

(c) $-3a$ (d) $2a$

16. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then

(a) $3bc - 2ad = 0$ *(b)* $3bc + 2ad = 0$
(c) $2bc - 3ad = 0$ *(d)* $2bc + 3ad = 0$

17. For $a > 0$, $b > 0$, $c > 0$, if the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then,

(a) $a + b - c > 0$ *(b)* $a - b + c < 0$
(c) $a - b + c > 0$ *(d)* $a + b - c < 0$

18. The locus of the image of the point $(2, 3)$ in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

(a) circle of radius $\sqrt{2}$
(b) circle of radius $\sqrt{3}$
(c) straight line parallel to X-axis
(d) straight line parallel to Y-axis

19. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d_1(P) + d_2(P) \le 4$, is

(a) 4 sq units *(b)* 6 sq units
(c) 8 sq units *(d)* 2 sq units

20. The area of the triangle formed by the lines $y = x$, $y = 0$ and $x = \sin^{-1}(a^4 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^4 + 1)$ is

(a) $\left(\frac{\pi^2}{8} - \frac{a^2}{4}\right)$ sq unit *(b)* $\left(\frac{\pi^2}{8} - \frac{a^2}{2}\right)$ sq unit
(c) $\frac{\pi^2}{16}$ sq unit *(d)* $\frac{\pi^2}{32}$ sq unit

21. If the point (a, a) falls between the line $|x + y| = 2$, then

(a) $|a| = 2$ *(b)* $|a| = 1$ *(c)* $|a| < 1$ *(d)* $|a| < \frac{1}{2}$

22. ABC is an isosceles triangle in which A is $(-1, 0)$, $\angle A = 2\pi / 3$, $AB = AC$ and AB is along the X-axis. If $BC = 4\sqrt{3}$, then the equation of the line BC is

(a) $x + \sqrt{3}y - 3$ *(b)* $\sqrt{3}x + y = 3$
(c) $x + y = \sqrt{3}$ *(d)* None of these

23. Point $P(2, 4)$ is translated through a distance $3\sqrt{2}$ units measured parallel to the line $y - x - 1 = 0$, in the direction of decreasing ordinates, to reach at Q. If R is the image of Q with respect to the line $y - x - 1 = 0$, then the coordinates of R are

(a) $(-1, 1)$ *(b)* $(5, 7)$ *(c)* $(6, 6)$ *(d)* $(0, 0)$

24. If the distance of any point (x, y) from the origin is defined as $d(x, y) = \max[|x|, |y|]$, then the locus of the point (x, y), where $d(x, y) = 1$, is a

(a) circle *(b)* square
(c) triangle *(d)* None of these

25. A family of lines is given by $(1 + 2\lambda)x + (1 - \lambda)y + \lambda = 0$, λ being the parameter. The line belonging to this family at the maximum distance from the point $(1, 4)$, is

(a) $4x - y + 1 = 0$ *(b)* $33x + 12y + 7 = 0$
(c) $12x + 33y = 7$ *(d)* None of these

26. The coordinates of the four vertices of a quadrilateral are $(-2, 4)$, $(-1, 2)$, $(1, 2)$ and $(2, 4)$ taken in order. The equation of the line passing through the vertex $(-1, 2)$ and dividing the quadrilateral in two equal areas, is

(a) $x + 1 = 0$ *(b)* $x + y = 1$
(c) $x - y + 3 = 0$ *(d)* None of these

27. If the roots of the equation $ax^2 + bx + 10 = 0$ are not real and distinct, where $a, b \in R$ and α, β are the values of a, b for which $5a + b$ is minimum, then the family of lines $\alpha(4x + 2y + 3) + \beta(x - y - 1) = 0$ are concurrent at

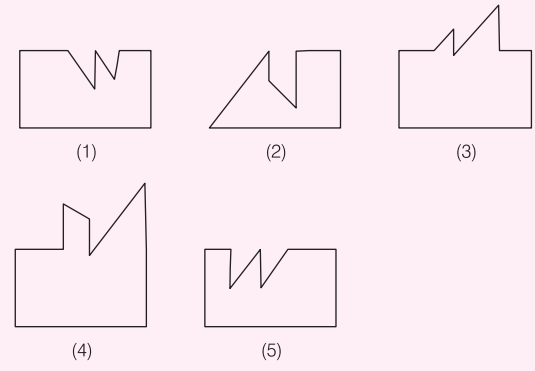
(a) $(1, -1)$ *(b)* $(1, 1)$
(c) $(-1/6, -7/6)$ *(d)* $(1/6, 7/6)$

Answers

1. (c)	**2.** (c)	**3.** (c)	**4.** (c)	**5.** (a)
6. (b)	**7.** (a)	**8.** (a)	**9.** (b)	**10.** (b)
11. (a)	**12.** (d)	**13.** (b)	**14.** (d)	**15.** (b)
16. (a)	**17.** (a)	**18.** (a)	**19.** (b)	**20.** (d)
21. (c)	**22.** (a)	**23.** (d)	**24.** (b)	**25.** (c)
26. (c)	**27.** (c)			

Figure out the Odd One !!!

In the following question, five figures marked 1, 2, 3, 4 and 5 are given. From these five figures, we can get two pairs of figures such that each pair forms a complete square. You have to select the odd figure which does not fit into any of the other figures to form a complete square.



PAIR OF STRAIGHT LINE

Homogeneous Equation

An equation of the form $a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \ldots + a_nx^n = 0$ in which the sum of the powers of x and y in every term is same (i.e. n), is called a homogeneous equation (of degree n).

Now, let us look at the following facts

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through origin.
  (a) If $h^2 > ab$, then lines are distinct and real.
  (b) If $h^2 = ab$, then lines are coincident.
  (c) If $h^2 < ab$, then lines are imaginary.

(ii) If $y = m_1x$ and $y = m_2x$ are two equations represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines $ax^2 + 2hxy + by^2 = 0$, then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

Note (i) Lines are right angle to each other, if $a + b = 0$, i.e. coefficient of x^2 + coefficient of $y^2 = 0$.
(ii) Equally inclined to the X-axis is $h = 0$, i.e. coefficient of $xy = 0$.
(iii) A homogeneous equation of degree n represents n straight lines passing through the origin.

General Equation of Second Degree Representing a Pair of Straight Line

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(i) Angle between the two lines representing by a general equation is $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$.

(ii) The joint equation of pair of straight lines joining origin to the points of intersection of the line given by $lx + my + n = 0$ and second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx + my}{-n}\right) + 2fy\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

Above equation is obtained by homogeneous equation of second degree curve with the help of $\left(\frac{lx + my}{-n}\right) = 1$.

(iii) The equation to the straight lines bisecting the angle between the straight lines $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

(iv) The product of perpendiculars drawn from (x_1, y_1) to the pair of straight lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}}$.

SPEED BOOSTER

- The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent two parallel lines, if $g^2 - ac > 0$ and $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ and the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a + b)}}$ or $2\sqrt{\frac{f^2 - ac}{b(a + b)}}$.
- If the equation of a pair of straight line is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the point of intersection is given by $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

MASTER STROKES

1. The orthocentre of the triangle formed by the pair of lines $2x^2 - xy - y^2 + x + 2y - 1 = 0$ and the line $x + y + 1 = 0$, is
(a) $(-1, 0)$ *(b)* $(0, 1)$
(c) $(-1, 1)$ *(d)* None of these

2. If θ is the angle between the straight lines given by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$, then $\text{cosec}^2\theta$ equals
(a) 9 *(b)* 10 *(c)* 15 *(d)* 26

3. The angle between the pair of lines whose equation is $4x^2 + 10xy + my^2 + 5x + 10y = 0$, is
(a) $\tan^{-1}\frac{3}{8}$
(b) $\tan^{-1}\frac{3}{4}$
(c) $\tan^{-1}\left(\frac{2\sqrt{24 - 4m}}{m + 4}\right)$, $m \in R$
(d) None of the above

4. If one of the lines represented by the equation $\lambda y^2 + (1-\lambda)xy - \lambda x^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then λ equals

(a) $1, -1$ (b) $2, -\frac{1}{2}$

(c) $1, 2$ (d) $-1, -\frac{1}{2}$

5. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of 45° at the major segment of the circle, then m equals

(a) ± 2 (b) $1, 2$

(c) ± 1 (d) None of these

6. If two pairs of straight lines having equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ have one line common, then a equals

(a) $-3(2h + 3b)$ (b) $8(h - 2b)$

(c) $2(b + h)$ (d) $-3(b + h)$

7. The product of the perpendiculars drawn from the point (1, 2) to the pair of lines $x^2 + 4xy + y^2 = 0$, is

(a) $\frac{9}{4}$ (b) $\frac{3}{4}$

(c) $\frac{9}{16}$ (d) None of these

8. The equation $2x^2 - 3xy - py^2 + x + qy - 1 = 0$ represents two mutually perpendicular lines, if

(a) $p = 3, q = 2$ (b) $p = 2, q = 3$

(c) $p = -2, q = 3$ (d) $p = 2, q = -\frac{9}{2}$

9. If the equation $\lambda x^2 + 4xy + 5y^2 = 0$ represents two lines inclined at an angle π, then λ equals

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$

(c) -45 (d) None of these

10. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the line $xy = 0$, then m equals

(a) 1 (b) 2

(c) $-\frac{1}{2}$ (d) -2

11. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between positive direction of the axes, then a, b and h satisfy the relation

(a) $a + b = 2|h|$ (b) $a + b = -2h$

(c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$

12. Let PQR be a right angled isosceles triangle, right angled at P (2, 1). If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is

(a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$

(b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$

(d) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$

13. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents a pair of perpendicular straight lines, then

(a) $p - 12, q = -1$ (b) $p = -12, q = 1$

(c) $p = 12, q = 1$ (d) $p = 1, q = 12$

14. The equation $x^3 + y^3 = 0$ represents

(a) three real straight lines

(b) three points

(c) the combined equation of a straight line and a circle

(d) None of the above

15. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals

(a) -3 (b) -1

(c) 3 (d) 1

16. The square of the distance between the origin and the point of intersection of the lines given by $ax^2 + 2hxy + by^2 + 2qx + 2fy + c = 0$, is

(a) $\frac{c(a+b) + f^2 + g^2}{ab - h^2}$ (b) $\frac{c(a+b) - f^2 - g^2}{h^2 - ab}$

(c) $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$ (d) None of these

17. The equation $\lambda(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines passing through the origin such that

(a) they are equally inclined to one another

(b) two of which are at right angles

(c) two of which are coincident

(d) None of the above

18. If the lines represented by $x^2 + 2\lambda xy + 2y^2 = 0$ and the lines represented by $(1 + \lambda)x^2 - 8xy + y^2 = 0$ are equally inclined, then

(a) λ is any real number (b) $\lambda > 2$

(c) $\lambda = \pm 2$ (d) $\lambda < -2$

19. The area of a triangle formed by the line $x + y = 3$ and angle bisectors of the pair of straight line $x^2 - y^2 + 2y = 1$, is

(a) 2 sq units (b) 4 sq units

(c) 6 sq units (d) 8 sq units

20. The straight line represented by $x^2 + mxy - 2y^2 + 3y - 1 = 0$ meet at the point

(a) $\left(-\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(-\frac{1}{3}, -\frac{2}{3}\right)$

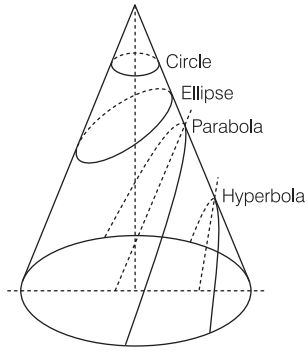
(c) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (d) None of these

Answers

1. (a)	**2.** (b)	**3.** (b)	**4.** (a)	**5.** (c)
6. (a, b)	**7.** (d)	**8.** (b,d)	**9.** (b)	**10.** (a)
11. (b)	**12.** (b)	**13.** (c)	**14.** (d)	**15.** (a)
16. (c)	**17.** (a)	**18.** (c)	**19.** (a)	**20.** (a,c)

CIRCLE

A conic section or conic is the locus of a point which moves in a plane such that its distance from a fixed point (i.e. focus) always bears a constant ratio to its distance from a fixed line (i.e. directrix). This constant ratio is called eccentricity of the conic and it is denoted by 'e'.



General Equation of Conic

A second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents degenerate and non-degenerate conic section.

1. Degenerate Conic Section

When the focus lies on the directrix, then above equation represents pair of straight lines, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Now, we have the following cases:

(i) If $e > 1$ or $h^2 > ab$, then the lines will be real and distinct intersecting at fixed point.

(ii) If $e = 1$ or $h^2 = ab$, then the lines will be coincident (parallel).

(iii) If $e < 1$ or $h^2 < ab$, then the lines will be imaginary.

2. Non-degenerate Conic Section

When the focus does not lie on the directrix, then above equation represent

(i) Circle : $a = b$, $h = 0$ and $\Delta \neq 0$ or $e = 0$.

(ii) Parabola : $h^2 = ab$ and $\Delta \neq 0$ or $e = 1$.

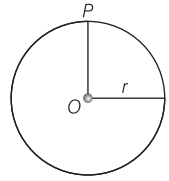
(iii) Ellipse : $h^2 < ab$ and $\Delta \neq 0$ or $0 < e < 1$.

(iv) Hyperbola : $h^2 > ab$ and $\Delta \neq 0$ or $e > 1$.

(v) Rectangular hyperbola : $a + b = 0$, $\Delta \neq 0$ and $h^2 > ab$.

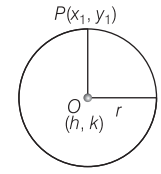
Circle as a Locus

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point always remains constant. This fixed point is called centre of circle and the constant distance is called radius of circle.



Equation of Circle in Various Form

1. Centre-Radius Form



If $P(x_1, y_1)$ is any point on the circle and r is the radius of the circle, then

(i) Equation of a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

(ii) Equation of a circle with centre $(0, 0)$, i.e. at origin and radius r is $x^2 + y^2 = r^2$.

2. Parametric Form

The parametric equation of the circle $x^2 + y^2 = a^2$ is

$x = a\cos\theta$ and $y = a\sin\theta$, where θ is a parameter.

Note (i) The parametric equations of a circle $(x - h)^2 + (y - k)^2 = a^2$ is

$x = h + a\cos\theta$ and $y = k + a\sin\theta$

(ii) The equation of the chord of a circle $x^2 + y^2 = a^2$ joining $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ is

$$x\cos\left(\frac{\alpha + \beta}{2}\right) + y\sin\left(\frac{\alpha + \beta}{2}\right) = a\cos\left(\frac{\alpha - \beta}{2}\right).$$

3. General Form

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where g, f and c are constants, $(-g, -f)$ is centre and $r = \sqrt{g^2 + f^2 - c}$ is the radius of a circle.

REMEMBER

- Nature of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ has following possible cases:
  (a) If $g^2 + f^2 - c > 0$, then real circle is possible.
  (b) If $g^2 + f^2 - c = 0$, then radius of the circle is zero. Hence, circle is called point circle.
  (c) If $g^2 + f^2 - c < 0$, then radius of the circle will be imaginary number. Hence, in this case, circle is called imaginary circle.
- Two circles having same centre but different radii are called concentric circles. Hence, equation of concentric circle differ by only radius. If (h, k) is centre and r_1, r_2 are radii of the concentric circles, then $(x - h)^2 + (y - k)^2 = r_1^2$ and $(x - h)^2 + (y - k)^2 = r_2^2$.
- To find centre and radius of the circle, use following steps:
  ***Step* I** Make the coefficients of x^2 and y^2 equal to 1 and right hand side equal to zero.
  ***Step* II** Coordinates of centre $(-g, -f)$, where
  $g = \frac{1}{2}$(Coefficient of x) and $f = \frac{1}{2}$(Coefficient of y)
  and radius is $\sqrt{g^2 + f^2 - \text{Constant term}}$.

4. Diametric Form

The equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as its diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Equation of Circle Passing through Three Non-collinear Points

If three non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then their coordinates satisfy its equation.

i.e. $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$

and $x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$

On solving these equations, we can obtain the values of g, f and c and put these values in the equation of circle to get the required equation of circle.

Intercept Made by a Circle on the Axes

The intercepts made by a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) on X-axis is $2\sqrt{g^2 - c}$, where $g^2 > c$.

(ii) on Y-axis is $2\sqrt{f^2 - c}$, where $f^2 > c$.

REMEMBER

- If $g^2 > c \Rightarrow$ Circle cuts the X-axis at two distinct points.
- If $g^2 = c \Rightarrow$ Circle touches the X-axis.
- If $g^2 < c \Rightarrow$ Circle lies completely above or below the X-axis.

  Similar in the case for intercepts on Y-axis.

Position of a Point with Respect to a Circle

Let equation of a circle be $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

If value of a circle at $P(x_1, y_1)$ is $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then following cases arise:

(i) If $S_1 = 0$, then point lies on the circle.

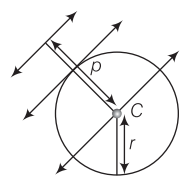
(ii) If $S_1 > 0$, then point lies outside the circle.

(iii) If $S_1 < 0$, then point lies inside the circle.

Note Maximum and minimum distance of the point P from the circle is $|r + PC|$ and $|r - PC|$, where C and r are the centre and radius of a circle, respectively.

Line and Circle

Let $L = 0$ and $S = 0$ be the equations of line and circle, respectively. If r is the radius and p is the length of the perpendicular from the centre of circle to the line $L = 0$, then the following cases arise:



(i) If $p > r$, then the line does not meet the circle, i.e. it passes outside.

(ii) If $p = r$, then the line touches the circle.

(iii) If $p < r$, then the line is secant to the circle.

(iv) If $p = 0$, then the line is a diameter of the circle.

Tangent to a Circle

If a line touches the circle at a point, then the line is called tangent to the circle and the point, where it touches the circle, is called the point of contact.

Various Form of the Equation of Tangent

1. Cartesian Form

If (x_1, y_1) is the point of contact, then the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Note If (x_1, y_1) is the point of contact, then the equation of tangent to the circle $(x - h)^2 + (y - k)^2 = r^2$ is $(x - h)(x - x_1) + (y - k)(y - y_1) = r^2$.

2. Parametric Form

If $(r\cos\theta, r\sin\theta)$ are the parametric coordinates of the circle $x^2 + y^2 = r^2$, then the equation of tangent is $x\cos\theta + y\sin\theta = r$.

Note The point of intersection of tangents drawn to the circle at the points $P(\theta)$ and $Q(\alpha)$ is $\left[\dfrac{r\cos\left(\frac{\theta+\alpha}{2}\right)}{\cos\left(\frac{\theta-\alpha}{2}\right)}, \dfrac{r\sin\left(\frac{\theta+\alpha}{2}\right)}{\cos\left(\frac{\theta-\alpha}{2}\right)}\right]$.

3. Slope Form

The equation of a tangent of slope m to the circle $x^2 + y^2 = r^2$ is $y = mx \pm r\sqrt{1 + m^2}$.

Note The condition that a line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$, is $c = \pm r\sqrt{1 + m^2}$ and point of contact is $\left(\pm\dfrac{r^2 m}{c}, \mp\dfrac{r^2}{c}\right)$.

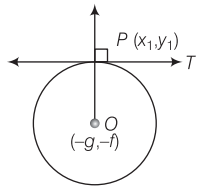
REMEMBER

- If slope of the tangent is given, then two parallel tangents can be drawn to the circle at the end of diameter.
- Equation of tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of slope is

  $$y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}.$$
- If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then the point of contact is given by $\left(-\dfrac{ar^2}{c}, \dfrac{-br^2}{c}\right)$.

Normal to a Circle

The normal of a circle at any point is a straight line which is perpendicular to the tangent line at the point of contact.



Note Normal to a circle always passes through the centre.

Various Form of the Equation of Normal to a Circle

1. Point Form

If $P(x_1, y_1)$ is the point of contact, then the equation of normal to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

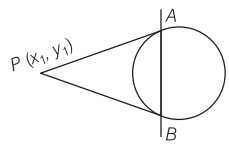
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1) \quad \text{or} \quad \left(\frac{y - y_1}{y_1 + f}\right) = \left(\frac{x - x_1}{x_1 + g}\right)$$

2. Parametric Form

If $(r\cos\theta, r\sin\theta)$ are the parametric coordinates of a circle $x^2 + y^2 = r^2$, then the equation of normal at $(r\cos\theta, r\sin\theta)$ is $y = x\tan\theta$.

Equation of Pair of Tangent

The equation of pair of tangents drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $T^2 = SS_1$,

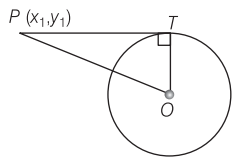


where $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

and $T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Length of the Tangent

If $P(x_1, y_1)$ is any point lies outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then length of the tangent is $PT = \sqrt{S_1}$



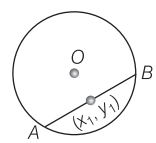
where,

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note To find the length of the tangent from a point, make the coefficient of x^2 and y^2 equal to 1.

Equation of Chord Bisected at a Given Point

If $P(x_1, y_1)$ bisects the chord AB of a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of chord AB is $T = S_1$



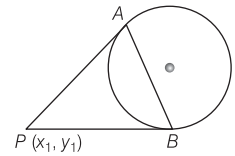
where,

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

and $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Equation of Chord of Contact

If two tangents AP and BP are drawn from an external point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact AB is $T = 0$



where, $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

REMEMBER

- The length of chord of contact is $2\sqrt{r^2 - p^2}$, where p is length of perpendicular from centre to the chord and r is radius of the circle.
- If $S = x^2 + y^2 - r^2$, then area of ΔABP is given by $\frac{r(x_1^2 + y_1^2 - r^2)^{3/2}}{x_1^2 + y_1^2}$ or $\frac{r S_1\sqrt{S_1}}{(S_1 + r^2)}$.

Diameter

The locus of the middle points of a system of parallel chords of a conic is called a diameter of the circle.

The equation of the diameter bisecting parallel chord $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = r^2$ is $x + my = 0$.

Common Tangents to Two Circles

Two circles may intersect, touch or neither intersect nor touch according to the following condition, which are given below :

Let the centres and radii of two circles be C_1, C_2 and r_1, r_2 respectively.

(i) Circles neither intersect nor touch each other, when $C_1C_2 > r_1 + r_2$. In this case, there are four common tangents. Two of them are Transverse Common Tangent (TCT) and other two are Direct Common Tangent (DCT).

(ii) Two circles intersect each other when $|r_1 - r_2| < C_1C_2 < |r_1 + r_2|$. In this case, there are two common tangents and both of them are DCT.

(iii) Two circles touch each other

(a) externally when $C_1C_2 = r_1 + r_2$. In this case, there are three common tangents possible. Two of them are DCT and one is TCT.

(b) internally when $C_1C_2 = r_1 - r_2$. In this case, one common tangent is possible, which is DCT.

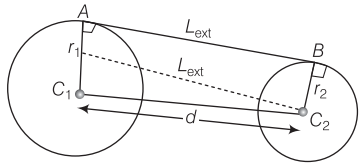
Note (i) DCT meets at a point which divides the line joining centre of circles externally in the ratio of their radii.

(ii) TCT meets at a point which divides the line joining centre of circles internally in the ratio of their radii.

Length of DCT and TCT

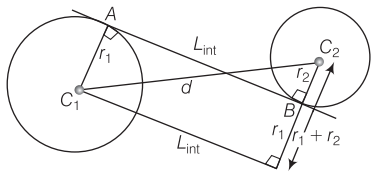
(i) If AB is external common tangent (DCT), then the length of external common tangent (or DCT) is

$$L_{ext} = L_{DCT} = \sqrt{d^2 - (r_1^2 - r_2^2)}$$



(ii) If AB is internal common tangent (TCT), then the length of internal common tangent (or TCT) is

$$L_{int} = L_{TCT} = \sqrt{d^2 - (r_1 + r_2)^2}$$



SPEED BOOSTER

- The length of tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{c - c_1}$.
- If the tangent to the circle $x^2 + y^2 = r^2$ at the point (a, b) meets the coordinate axes at the points A, B and O is the origin. Then, area of ΔAOB is $\frac{r^4}{2ab}$.
- If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and let P be the centre of the circle, then the area of the quadrilateral $OAPB$ is $\sqrt{c(g^2 + f^2 - c)}$.
- If the circles $x^2 + y^2 + 2gx + c^2 = 0$ and $x^2 + y^2 + 2fy + c^2 = 0$ touch each other, then $\frac{1}{g^2} + \frac{1}{f^2} = \frac{1}{c^2}$.
- The angle of intersection of two circles is defined as the angle between the tangents or angle between the normals to the two circles at their point of intersection, is given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$$

where, d is the distance between both the centres of the circle.

Common Chord

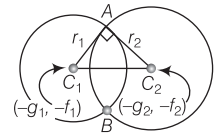
Let $S_1 \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$ be two given circles. Then, the chord joining the points of intersection of these circles is called their common chord and its equation is given by $S_1 - S_2 = 0$.

SPEED BOOSTER

- Length of common chord $= 2\sqrt{(C_1P)^2 - (C_1M)^2}$, where C_1P = radius of the circle $S_1 = 0$ and C_1M = length of the perpendicular from the centre C_1 to the common chord PQ.
- Length of common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$ is $\frac{2ab}{\sqrt{a^2 + b^2}}$.

Orthogonal Circle

Two circles $S_1 \equiv x^2 + y^2 + 2g_1 x + 2f_1 y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2 x + 2f_2 y + c_2 = 0$, are said to be orthogonal, if the tangents to the circles at each point of intersection are at 90°.



Note (i) The condition for two circles to be orthogonal is $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

(ii) If circles having radii r_1 and r_2 intersect orthogonally, then length of their common chord is $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$.

Director Circle

The locus of the point of intersection of two perpendicular tangents to a conic is called director circle.

The director circle of a circle is concentric circle, whose radius is $\sqrt{2}$ times the radius of the given circle.

Equation of Director Circle

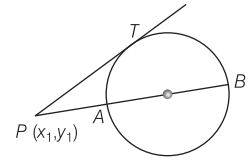
(i) If $x^2 + y^2 = r^2$ is the equation of circle, then equation of director circle is $x^2 + y^2 = 2r^2$.

(ii) If $x^2 + y^2 + 2gx + 2fy + c$ is the equation of the circle, then equation of director circle is

$$x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$$

Power of Point

If $P(x_1, y_1)$ is any point outside the circle, then the power of point $P(x_1, y_1)$ with respect to circle is $PA \cdot PB$.



$\because\ PA \cdot PB = PT^2 = (\sqrt{S_1})^2$

Hence, power of point to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the tangent segments drawn from it to the two circles are of equal length.

Equation of Radical Axis

The equation of radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$, i.e. $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$.

REMEMBER

- If two circles intersect or touch each other, then the radical axis is the common chord or common tangent of two circles, respectively.
- Radical axis is always perpendicular but need not always bisects the line joining the centres of two circles.
- Radical axis bisects a common tangent of two circles.
- The common point of intersection of the radical axes of three circles, whose centres are non-collinear, taken two at a time is called the radical centre of three circles.

Family of Circles

Important result related to family of circles are given below:

(i) If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ are two intersecting circles, then the equation of family of circles passing through the points of intersection of two given circles is $S + \lambda S_1 = 0$, where $\lambda \neq -1$.

(ii) If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle which is intersected by a line $L \equiv ax + by + c = 0$ at two real and distinct points, then the equation of family of circles passing through the points of intersection of circle and line is $S + \lambda L = 0$.

(iii) The equation of family of circles passing through (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

where λ is a parameter.

(iv) The equation of family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + k[y - y_1 - m(x - x_1)] = 0$.

Two particular cases are as follow:

(a) If line passes through (x_1, y_1) and parallel to Y-axis, then $(x - x_1)^2 + (y - y_1)^2 + k(x - x_1) = 0$.

(b) If line passes through (x_1, y_1) and parallel to X-axis, then $(x - x_1)^2 + (y - y_1)^2 + k(y - y_1) = 0$.

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- Equation of circle circumscribing a triangle whose sides are $L_1 = 0, L_2 = 0$ and $L_3 = 0$, is given by $L_1L_2 + k_1L_2L_3 + k_2L_3L_1 = 0$ provided, coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.
- Equation of circle circumscribing a quadrilateral whose sides in order are represented by $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$, is $L_1L_3 + kL_2L_4 = 0$ provided, coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

Coaxial System of Circles

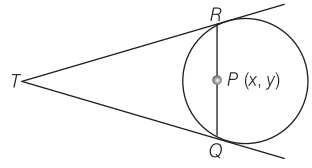
A system of circles is said to be coaxial system of circles, if every pair of the circles in the system has the same radical axis. Now, let us consider the following cases:

***Case* I** If the equation of a member of a system of coaxial circles is $S = 0$ and the equation of the common radical axis is $L = 0$, then the equation representing the coaxial system of circle is $S + \lambda L = 0$, where $\lambda \in R$.

***Case* II** The equation of a coaxial system of circles of which two members are $S_1 = 0$ and $S_2 = 0$, is $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$ or $S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \lambda(S_1 - S_2) = 0$, where $\lambda \in R$.

Pole and Polar

Let P be a point lying within or outside a given conic. Suppose any straight line drawn through P intersects the conic at Q and R. Then, the locus of the point of intersection of the tangents to the conic at Q and R is called the polar of the given point P with respect to the given conic and the point P is called the pole of the polar. The polar of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$ or $T = 0$.



1. Conjugate Points

Two points A and B are conjugate points with respect to a given conic, if each lies on the polar of the other.

2. Conjugate Lines

If two lines is such that the pole of one line lies on the other, then they are called conjugate lines with respect to the given conic.

Brain Shaker!!!

Each family in a locality has atmost two adults and no family has fewer than 3 children. Considering all the families together, there are more adults than boys, more boys than girls and more girls than families. Then, what will be the minimum possible number of families in the locality?

MASTER STROKES

1. Let $ABCD$ be a quadrilateral with area 18, side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 *(b)* 2 *(c)* 3/2 *(d)* 1

2. From a point $A(1, 1)$ on the circle $x^2 + y^2 - 4x - 4y + 6 = 0$, two equal chords AB and AC of length 2 units are drawn. The equation of chord BC is

(a) $4x + 3y - 12 = 0$ *(b)* $x + y = 4$
(c) $3x + 4y = 4$ *(d)* $x + y = 6$

3. C_1 and C_2 are the two concentric circles with radii r_1 and r_2 $(r_1 < r_2)$. If the tangents drawn from any point of C_2 to C_1 meet again C_2 at the ends of its diameter, then

(a) $r_2 = 2r_1$ *(b)* $r_2 = \sqrt{2}r_1$
(c) $r_2^2 < 2r_1^2$ *(d)* None of these

4. A circle C of radius 1 is inscribed in an equilateral ΔPQR. The points of contact of C with sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ. The equation of circle C is

(a) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$
(b) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$
(c) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$
(d) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

5. The tangents PA and PB are drawn from any point P of the circle $x^2 + y^2 = 2a^2$ to the circle $x^2 + y^2 = a^2$. The chord of contact AB on extending meets again the first circle at the points A' and B'. The locus of the point of intersection of tangents at A' and B' may be given as

(a) $x^2 + y^2 = 8a^2$ *(b)* $x^2 + y^2 = 4a^2$
(c) $x^2 + y^2 = 6a^2$ *(d)* None of these

6. If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 - px - qy = 0$ (where $pq \neq 0$) are bisected by the X-axis, then

(a) $p^2 = q^2$ *(b)* $p^2 = 8q^2$ *(c)* $p^2 < 8q^2$ *(d)* $p^2 > 8q^2$

7. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents

(a) four straight lines, when $c = 0$ and a, b are of the same sign
(b) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a
(c) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a
(d) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

8. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of ΔPAB as P moves on the circle, is a/an

(a) parabola
(b) circle
(c) ellipse
(d) pair of straight lines

9. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the Y-axis, then the length of PQ is

(a) 4 *(b)* $2\sqrt{5}$ *(c)* 5 *(d)* $3\sqrt{5}$

10. A circle circumscribing an equilateral triangle with centroid at (0, 0) of side 'a' is drawn and a square is drawn whose four sides touch the circle. The equation of the circle circumscribing the square is

(a) $x^2 + y^2 = 2a^2$ *(b)* $3x^2 + 3y^2 = 2a^2$
(c) $5x^2 + 5y^2 = 3a^2$ *(d)* None of these

11. If a circle is passing through the points $A(1, 1)$ and $B(1, 3)$ and the bisector of first and third quadrant is normal to it, then its area is

(a) 2π sq units *(b)* 4π sq units
(c) $\frac{\pi}{2}$ sq units *(d)* None of these

12. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at points A and B. The equation of the circumcircle of ΔPAB is

(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

13. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre (2, 1), then the radius of circle is

(a) $\sqrt{3}$ *(b)* $\sqrt{2}$ *(c)* 3 *(d)* 2

14. The centres of two circles C_1 and C_2 each of unit radii are at a distance of 6 units from each other. Let P be the mid-point of the line segment joining the centres of C_1, C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is

(a) 4 units *(b)* 8 units *(c)* 6 units *(d)* 3 units

15. In ΔABC, if equation of side BC is $x + y - 6 = 0$, also the circumcentre and orthocentre are $(3, 1)$ and $(2, 2)$ respectively, then the equation of the circumcircle of ΔABC is
(a) $x^2 + y^2 - 6x - 2y + 10 = 0$
(b) $x^2 + y^2 - 6x - 2y = 0$
(c) $x^2 + y^2 - 2x - 3y - 10 = 0$
(d) $x^2 + y^2 - 2x - 3y = 0$

16. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points, if
(a) $15 < m < 65$ *(b)* $35 < m < 85$
(c) $-85 < m < -35$ *(d)* $-35 < m < 15$

17. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \leq 6$ into two parts. If $S = \{(2, 3/4), (5/2, 3/4), (1/4, -1/4), (1/8, 1/4)\}$, then the number of points in S lying inside the smaller part, is
(a) 2 *(b)* 3
(c) 4 *(d)* 5

18. The circle passing through the point $(-1, 0)$ and touching the Y-axis at $(0, 2)$ also passes through the point
(a) $(-3/2, 0)$ *(b)* $(-5/2, 2)$
(c) $(-3/2, 5/2)$ *(d)* $(-4, 0)$

19. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$, is
(a) $20(x^2 + y^2) - 36x + 45y = 0$
(b) $20(x^2 + y^2) + 36x - 45y = 0$
(c) $36(x^2 + y^2) - 20x + 45y = 0$
(d) $36(x^2 + y^2) + 20x - 45y = 0$

20. A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$. A possible equation of L is
(a) $x - \sqrt{3}y = 1$ *(b)* $x + \sqrt{3}y = 1$
(c) $x - \sqrt{3}y = -1$ *(d)* $x + \sqrt{3}y = 5$

21. A common tangent to the circles $x^2 + y^2 = 4$ and $(x - 3)^2 + y^2 = 1$ is
(a) $x = 4$ *(b)* $y = 2$
(c) $x + \sqrt{3}y = 4$ *(d)* $x + 2\sqrt{2}y = 6$

22. If the line $y = mx + 1$ meets the circle $x^2 + y^2 + 3x = 0$ in two points equidistant from and on opposite sides of X-axis, then
(a) $3m - 2 = 0$ *(b)* $2m + 3 = 0$
(c) $3m + 2 = 0$ *(d)* $2m - 3 = 0$

23. If three distinct points A, B, C are given in the two-dimensional coordinate plane such that the ratio of the distance of each one of them from the point $(1, 0)$ to the distance from $(-1, 0)$ is equal to $1/2$, then the circumcentre of ΔABC is at the point
(a) $(3, 0)$ *(b)* $(5/3, 0)$
(c) $(1/3, 0)$ *(d)* $(0, 0)$

24. Circle(s) touching X-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on Y-axis, is/are
(a) $x^2 + y^2 - 6x \pm 8y + 9 = 0$
(b) $x^2 + y^2 - 6x \pm 7y + 9 = 0$
(c) $x^2 + y^2 + 6x \pm 8y + 9 = 0$
(d) $x^2 + y^2 - 8x \pm 6y + 9 = 0$

25. If a circle has two of its diameter along the lines $x + y = 5$ and $x - y = 1$ and has area 9π, then the equation of the circle is
(a) $x^2 + y^2 - 6x - 4y + 4 = 0$
(b) $x^2 + y^2 - 6x - 4y - 3 = 0$
(c) $x^2 + y^2 - 6x - 4y - 4 = 0$
(d) $x^2 + y^2 - 6x - 4y + 3 = 0$

26. Let C be the circle with centre at $(1, 1)$ and radius 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally, then the radius of T is equal to
(a) $\frac{1}{2}$ *(b)* $\frac{1}{4}$ *(c)* $\sqrt{\frac{3}{2}}$ *(d)* $\frac{\sqrt{3}}{2}$

27. If the common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the point R, S. Then, the area of the quadrilateral $PQRS$ is
(a) 3 sq units *(b)* 6 sq units
(c) 9 sq units *(d)* 15 sq units

28. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is
(a) 3 *(b)* 4 *(c)* 1 *(d)* 2

29. Tangents PA and PB are drawn to the circle $x^2 + y^2 = 8$ from any arbitrary point P on the line $x + y = 4$. The locus of the mid-point of chord of contact AB, is
(a) $x^2 + y^2 + 2x + 2y = 0$ *(b)* $x^2 + y^2 - 2x - 2y = 0$
(c) $x^2 + y^2 - 2x + 2y = 0$ *(d)* $x^2 + y^2 + 2x - 2y = 0$

30. If two circles are $x^2 + y^2 + 3\sqrt{2}(x + y) = 0$ and $x^2 + y^2 + 5\sqrt{2}(x + y) = 0$ and the radius of a circle which touches the two circles and their common diameter is $\frac{2\lambda - 1}{\lambda}$, then λ equals
(a) 10 *(b)* 8 *(c)* 7 *(d)* 5

Answers

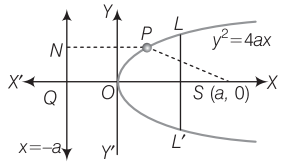
1. (b)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (a)
6. (d)	**7.** (b)	**8.** (b)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (c)	**14.** (b)	**15.** (b)
16. (d)	**17.** (a)	**18.** (d)	**19.** (a)	**20.** (a)
21. (d)	**22.** (b)	**23.** (b)	**24.** (a)	**25.** (a)
26. (b)	**27.** (d)	**28.** (a)	**29.** (b)	**30.** (b)

PARABOLA

A parabola is the locus of a point which moves in a plane in such a way that its distance from a fixed point (i.e. focus) is equal to its distance from a fixed straight line (i.e. directrix).

Standard Equation of Parabola

Let S and QN be the focus and directrix of the parabola, respectively. If P is any point on the parabola and $PN \perp QN$, then by definition



$$PS = PN \Rightarrow (PS)^2 = (PN)^2$$

[on squaring both sides]

$$\Rightarrow (x-a)^2 + y^2 = (x+a)^2 \Rightarrow y^2 = 4ax$$

which is the required equation of parabola.

Parametric Equations of Parabola

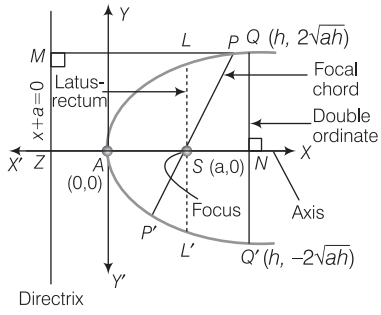
The parametric equations of parabola $y^2 = 4ax$ are $x = at^2$ and $y = 2at$, where t is the parameter.

Some Important Terms of Parabola

(i) **Axis** The axis of the parabola is the straight line which passes through focus and perpendicular to the directrix of the parabola.
For parabola $y^2 = 4ax$, X-axis is the axis of parabola.

(ii) **Symmetry** On replacing y by $(-y)$, then the equation $y^2 = 4ax$ remains same. So, the curve is symmetrical about X-axis.

(iii) **Vertex** The point of intersection of parabola and its axis is called vertex of parabola.



For parabola $y^2 = 4ax$, $A(0, 0)$ (i.e. origin) is the vertex.

(iv) **Ordinate and double ordinate** Let Q be any point on the parabola. From Q, draw $QN \perp AX$, then QN is called ordinate and QNQ' is called double ordinate of the point Q.

(v) **Focal chord** A chord of a parabola passing through its focus is called focal chord.

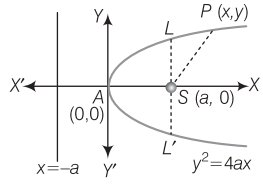
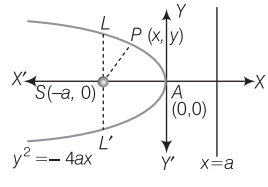
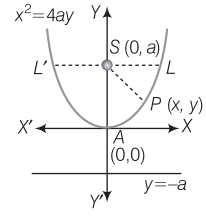
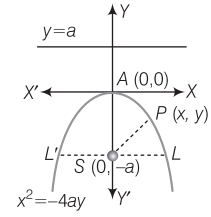
(vi) **Latusrectum** The focal chord perpendicular to the axis is called latusrectum.
If LL' is the latusrectum of a parabola, then this line is perpendicular to AX and passing through S. Also, $L \equiv (a, 2a)$, $L' \equiv (a, -2a)$ and length of latusrectum $= LL' = 4a$

(vii) **Focal distance** The distance of any point $P(x, y)$ from its focus S is known as its focal distance (SP).
According to the definition of parabola,

$$SP = PM = x + a$$

Standard Forms of the Parabola

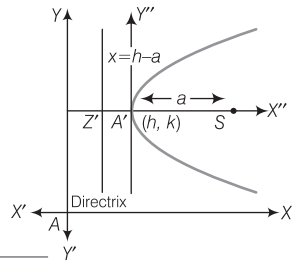
Four standard forms of the parabola are shown below alongwith their properties.

Standard equations	$y^2 = 4ax\ (a > 0)$	$y^2 = -4ax\ (a > 0)$	$x^2 = 4ay\ (a > 0)$	$x^2 = -4ay\ (a > 0)$
Shape of the parabola				
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latusrectum	$4a$	$4a$	$4a$	$4a$
Equation of latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity	1	1	1	1

Generalised Form of a Parabola

When origin is shifted at $A'(h, k)$ without changing the direction of the axes, then the equation of a parabola is

$$(y-k)^2 = 4a(x-h)$$



REMEMBER

- Axis $A'X'' \parallel AX$ with its vertex at $A'(h, k)$.
- The focus and length of latusrectum of parabola $(y-k)^2 = 4a(x-h)$ are $(a+h, k)$ and $4a$, respectively.
- The equation of the directrix of parabola $(y-k)^2 = 4a(x-h)$ is $x + a - h = 0$.
- Parametric equations of parabola $(y-k)^2 = 4a(x-h)$ are $x = h + at^2$ and $y = k + 2at$.

Various Forms of the Equation of Chord

1. Point form

The equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is $y(y_1 + y_2) = 4ax + y_1y_2$.

2. Parametric form

The equation of chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2(x + at_1t_2)$.

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- Length of the chord $y = mx + c$ to the parabola $y^2 = 4ax$ is $\frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$.
- The length of the chord joining two points t_1 and t_2 on the parabola $y^2 = 4ax$ is $a(t_1 - t_2)\sqrt{(t_1+t_2)^2 + 4}$.

Condition for the Chord to be a Focal Chord

The chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is focal chord iff $t_1t_2 = -1$.

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- The length of the focal chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $a(t_2 - t_1)^2$.
- The length of the focal chord passing through the point t on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.
- If one extremity of a focal chord is $(at_1^2, 2at_1)$, then the other extremity $(at_2^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ by virtue of relation $t_1t_2 = -1$.
- The focal chord of parabola $y^2 = 4ax$ making an angle α with the X-axis is of length $4a\,\text{cosec}^2\alpha$ and perpendicular on it from the vertex is $a \sin\alpha$.
- The length of a focal chord of a parabola varies inversely as the square of its distance from the vertex.
- If l_1 and l_2 are the length of segments of a focal chord of a parabola, then its latusrectum is $\frac{4l_1l_2}{l_1 + l_2}$.

Various Forms of the Equation of Tangent

1. Point Form

The equation of tangents to all standard forms of parabola at point (x_1, y_1) are given below:

Equations of parabola	Equations of tangent
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

2. Parametric Form

The parametric equations of tangents to all standard forms of parabola are given below:

Equations of the parabola	Points of contact	Equations of tangent at t
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

3. Slope Form

The equation of tangents and condition of tangency of a line in terms of slope (m) are given below:

Parabola	Equations of tangent in terms of slope (m)	Points of contact	Condition of tangency of line $y = mx + c$
$y^2 = 4ax$	$y = mx + \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$c = \frac{a}{m}$
$y^2 = -4ax$	$y = mx - \frac{a}{m}$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$y = mx - am^2$	$(2am, am^2)$	$c = -am^2$
$x^2 = -4ay$	$y = mx + am^2$	$(-2am, -am^2)$	$c = am^2$

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- The tangent at any point P on the parabola bisects the angle between the focal chord through P and the perpendicular from P on the directrix.
- The portion of a tangent to a parabola cut-off between the directrix and the curve subtends a right angle at the focus.
- Tangents at the extremities of any focal chord intersect at right angle on the directrix.
- Any tangent to a parabola and the perpendicular on it from the focus, meets on the tangent at the vertex.
- Circle circumscribing the triangle formed by any tangent, normal and X-axis, has its centre at focus.

Various Forms of the Equation of Normal

1. Point Form

The equation of normals to all standard forms of parabola at point (x_1, y_1) are given below:

Equations of parabola	Equations of normal
$y^2 = 4ax$	$y - y_1 = \frac{-y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 = \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 = -\frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 = \frac{2a}{x_1}(x - x_1)$

2. Parametric Form

The parametric equations of normals to all standard forms of parabola are given below:

Equations of parabola	Parametric coordinates	Equations of normals at 't'
$y^2 = 4ax$	$(at^2, 2at)$	$y + tx = 2at + at^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y - tx = 2at + at^3$
$x^2 = 4ay$	$(2at, at^2)$	$x + ty = 2at + at^3$
$x^2 = -4ay$	$(2at, -at^2)$	$x - ty = 2at + at^3$

3. Slope Form

The equation of normals and condition of normality of a line in terms of slope (m), are given below:

Equations of parabola	Point of contact in terms of slope (m)	Equations of normal in terms of slope (m)	Condition of normality of line $y = mx + c$
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx - 2am - am^3$	$c = -2am - am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx + 2am + am^3$	$c = 2am + am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$	$c = 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$	$c = -2a - \frac{a}{m^2}$

REMEMBER

- The line $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, if $al(l^2 + 2m^2) + m^2 n = 0$.
- If the normals at points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meets parabola $y^2 = 4ax$ at the same point, then $t_1 t_2 = 2$.
- If the normal at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.
- If the normal to a parabola $y^2 = 4ax$, makes an angle ϕ with the axis, then it will cut the curve again at an angle $\tan^{-1}\left(\frac{1}{2}\tan\phi\right)$.
- If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of the ordinate of P and Q is $8a^2$.

Conormal Points

Any three points on the parabola at which the normals pass through a common point are called conormal points and these points are also known as the feets of the normals.

If three normals are drawn through a common point, say (h, k) to the parabola $y^2 = 4ax$, then their slopes are the roots of the cubic equation $k = mh - 2am - am^3$.

REMEMBER

- The sum of the slopes of the normals at conormal points is zero, i.e. $m_1 + m_2 + m_3 = 0$.
- The sum of the ordinates of the conormal points is zero. For the parabola $y^2 = 4ax$, we have
  $$-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0.$$
- The centroid of the triangle formed by the conormal points lies on the axis of the parabola. If the vertices of the triangle formed by the conormal points are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$, then y-coordinate of the centroid becomes
  $$\frac{-2a(m_1 + m_2 + m_3)}{3} = \frac{-2a}{3} \times 0 = 0.$$
  Hence, the centroid lies on the X-axis, i.e. axis of the parabola.
- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) are real, then $h > 2a$.

Position of Points with Respect to Parabola

Let equation of parabola be $S \equiv y^2 - 4ax = 0$.

If value of parabola at $P(x_1, y_1)$ is $S_1 \equiv y_1^2 - 4ax_1$, then the following cases arise:

(i) If $S_1 = 0$, then point lies on the parabola.
(ii) If $S_1 < 0$, then point lies inside the parabola.
(iii) If $S_1 > 0$, then point lies outside the parabola.

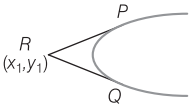
Number of Tangents Drawn from a Point to a Parabola

Two tangents can be drawn from a point to a parabola. The values of tangents are different, according to the position of a point, which are given as

(i) If a point lies outside, then two tangents are real and distinct.
(ii) If a point lies inside, then two tangents are imaginary.
(iii) If a point lies on the parabola, then two tangents are coincident.

Equation of Pair of Tangents

The equation of pair of tangents drawn from a point $R(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$, where $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.



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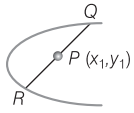
- Points of intersection of tangents at two points $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R\{at_1t_2, a(t_1 + t_2)\}$ (where, x-coordinate of R is GM of x-coordinate of P, Q and y-coordinate of R is AM of y-coordinate of P, Q.)
- Angle θ between tangents at two points $P(at_1^2, 2at_1), Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$, is given by $\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$.
- Locus of the point of intersection of perpendicular tangents to the parabola is directrix.

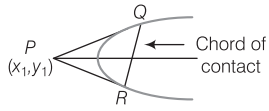
Equation of Chord Bisected at a Given Point

If any point $P(x_1, y_1)$ bisects the chord QR of the parabola $y^2 = 4ax$, then the equation of chord QR is $T = S_1$, where $T = yy_1 - 2a(x + x_1)$ and $S_1 = y_1^2 - 4ax_1$.



Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$



where, $T = yy_1 - 2a(x + x_1)$.

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- The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- If tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, then the length of their chord of contact is $\frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$.
- The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$.

Diameter

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

Director Circle

Equation of director circle of all parabolas are given below:

Equations of parabola	Equations of director circle
$y^2 = 4ax$	$x + a = 0$
$y^2 = -4ax$	$x - a = 0$
$x^2 = 4ay$	$y + a = 0$
$x^2 = -4ay$	$y - a = 0$

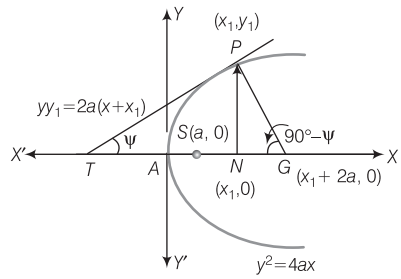
Note In case of parabola, director circle is directrix of the parabola.

Pole and Polar

The polar of a point (x_1, y_1) with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$ or $T = 0$.

Length of Tangent, Subtangent, Normal and Subnormal

Let the tangent and normal at any point $P(x_1, y_1)$ on a parabola $y^2 = 4ax$ meet the axis in T and G respectively, then PT = length of the tangent at P
PG = length of the normal at P
NT = length of subtangent at P
NG = length of subnormal at P



It can be seen from right angled ΔNPT and ΔPNG, that

(i) The length of tangent
$$= PT = PN \operatorname{cosec}\psi = y_1\sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$$

(ii) The length of the normal
$$= PG = PN \sec\psi = y_1\sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}}$$

(iii) The length of subtangent
$$= NT = PN \cot\psi = y_1\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

(iv) The length of the subnormal
$$= NG = PN \tan\psi = y_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

where, $\tan\psi = \frac{2a}{y_1} = m$

MASTER STROKES

1. A ray of light moving parallel to the X-axis gets reflected from parabolic mirror whose equation is $(y-3)^2 = 8(x+2)$. After reflection, the ray must pass through

(a) $(0, 3)$ *(b)* $(3, 0)$

(c) $(0, 0)$ *(d)* None of these

2. If two parabolas $y^2 = 4a(x-\lambda_1)$ and $x^2 = 4a(y-\lambda_2)$ always touch each other, λ_1 and λ_2 being variable parameters. Then, their points of contact lie on a

(a) straight line *(b)* circle

(c) parabola *(d)* hyperbola

3. The area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through $(1, 0)$ and having length $\frac{25}{4}$ units each, is

(a) $\frac{75}{4}$ sq units *(b)* $\frac{625}{16}$ sq units

(c) $\frac{25}{4}$ sq units *(d)* $\frac{25}{8}$ sq units

4. If (h, k) is a point on the axis of the parabola $2(x-1)^2 + (y-1)^2 = (x+y+2)^2$ from where three distinct normals may be drawn, then

(a) $h > 2$ *(b)* $h < 4$ *(c)* $h > 8$ *(d)* $h < 8$

5. A normal drawn at a point P on the parabola $y^2 = 4ax$ meets the curve again at Q. The least distance of Q from the axis of the parabola is

(a) $2\sqrt{2}a$ *(b)* $3\sqrt{2}a$

(c) $4\sqrt{2}a$ *(d)* None of these

6. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meets its axis at points T and N, respectively. The locus of the centroid of ΔPTN is a parabola whose

(a) vertex is $\left(\frac{2a}{3}, 0\right)$ *(b)* directrix is $x = 0$

(c) latusrectum is $\frac{2a}{3}$ *(d)* focus is $(-a, 0)$

7. The set of points on the axis of the parabola $y^2 = 4x + 8$ from which 3 normals to the parabola are real and distinct, is

(a) $\{(x, 0) : x \le -2\}$ *(b)* $\{(x, 0) : x > -2\}$

(c) $\{(0, y) : y > -2\}$ *(d)* None of these

8. Let A, B and C be three points taken on the parabola $y^2 = 4ax$ with coordinates $(at_i^2, 2at_i)$; $i = 1, 2, 3$; where t_1, t_2 and t_3 are in AP. If AA', BB' and CC' are focal chords and coordinates of A', B' and C' are $(at'^2_i, 2at'_i)$; $i = 1, 2, 3$; then t'_1, t'_2 and t'_3 are in

(a) AP *(b)* GP

(c) HP *(d)* None of these

9. The set of real values of a for which atleast one tangent to $y^2 = 4ax$ becomes normal to the circle $x^2 + y^2 - 2ax - 4ay + 3a^2 = 0$, is

(a) $[1, 2]$ *(b)* $[\sqrt{2}, 3]$

(c) R *(d)* None of these

10. Consider the two curves $C_1 : y^2 = 4x$ and $C_2 : x^2 + y^2 - 6x + 1 = 0$

Then,

(a) C_1 and C_2 touch each other at one point

(b) C_1 and C_2 touch each other exactly at two points

(c) C_1 and C_2 intersect (but do not touch) at exactly two points

(d) C_1 and C_2 neither intersect nor touch each other

11. The tangent and normal at the point $P(18, 12)$ of the parabola $y^2 = 8x$ intersects the X-axis at the points A and B respectively. The equation of the circle through P, A and B is given by

(a) $x^2 + y^2 + 4x - 540 = 0$

(b) $x^2 + y^2 - 6x - 360 = 0$

(c) $x^2 + y^2 - 4x - 396 = 0$

(d) $x^2 + y^2 - 2x - 444 = 0$

12. If the normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then

(a) $t_2 = t_1 + \frac{2}{t_1}$ *(b)* $t_2 = -t_1 - \frac{2}{t_1}$

(c) $t_2 = -t_1 + \frac{2}{t_1}$ *(d)* $t_2 = t_1 - \frac{2}{t_1}$

13. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the feets of the three normals drawn from a point to the parabola $y^2 = 4ax$, then $\frac{x_2 - x_3}{y_1} + \frac{x_3 - x_1}{y_2} + \frac{x_1 - x_2}{y_3}$ equals

(a) $4a$ *(b)* $2a$

(c) a *(d)* 0

14. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$, is

(a) $3y = 9x + 2$ *(b)* $y = 2x + 1$

(c) $2y = x + 8$ *(d)* $y = x + 2$

15. If normals at P, Q, R are drawn to $y^2 = 4x$ which intersect at $(3, 0)$, then area of ΔPQR is

(a) $\frac{2}{5}$ *(b)* $\frac{1}{2}$ *(c)* $\frac{5}{2}$ *(d)* 2

16. If normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at $(3, 0)$, then the circumcentre of ΔPQR has the coordinates

(a) $(2, 0)$ *(b)* $(1, 0)$

(c) $\left(\frac{2}{3}, 0\right)$ *(d)* $\left(\frac{5}{2}, 0\right)$

17. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$. Then, the locus of P is

(a) $x^2 = y$ (b) $y^2 = 2x$
(c) $y^2 = x$ (d) $x^2 = 2y$

18. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is not given by

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

19. Let S be the focus of the parabola $y^2 = 8x$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. Then, the area of ΔPQS is

(a) 4 sq units (b) 3 sq units
(c) 2 sq units (d) 8 sq units

20. Let PQ be a focal chord of the parabola $y^2 = 4ax$. If the tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$, then the length of chord PQ is

(a) $7a$ (b) $5a$
(c) $2a$ (d) $3a$

21. Let PQ be a focal chord of the parabola $y^2 = 4ax$ such that tangents at P and Q meet at a point on the line $y = 2x + a$, $a > 0$. If PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta$ equals

(a) $\frac{2\sqrt{7}}{3}$ (b) $-\frac{2\sqrt{7}}{3}$
(c) $\frac{2\sqrt{5}}{3}$ (d) $-\frac{2\sqrt{5}}{3}$

22. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$, is

(a) $\frac{1}{8}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

23. The locus of the mid-points of the chords of the parabola $x^2 = 4py$ having slope m, is a

(a) circle with centre at origin and radius $|2\ pm|$
(b) line parallel to X-axis at a distance $|2\ pm|$ from it
(c) line parallel to Y-axis at a distance $|2\ pm|$ from it
(d) line parallel to $y = mx$, $m \neq 0$ at a distance $|2\ pm|$ from it

24. Let a, r, s, t be non-zero real numbers and $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$, $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR, PK are parallel, where K is the point $(2a, 0)$. The value of r is

(a) $-\frac{1}{t}$ (b) $\frac{t^2 + 1}{t}$
(c) $\frac{1}{t}$ (d) $\frac{t^2 - 1}{t}$

25. In Q. 24, if $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(a) $\frac{(t+1)^2}{2t^3}$ (b) $\frac{a(t+1)^2}{2t^3}$
(c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$

26. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio $1:3$, then the locus of P is

(a) $y^2 = 2x$ (b) $x^2 = 2y$
(c) $x^2 = y$ (d) $y^2 = x$

27. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latusrectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is

(a) 4 (b) 2
(c) 6 (d) 9

28. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

(a) 3 (b) 6
(c) 8 (d) 4

29. If PSQ is a focal chord of a parabola whose focus is S and vertex is A. PA, QA are produced to meet the directrix in R and T. Then, $\angle RST$ is equal to

(a) $90°$ (b) $60°$
(c) $45°$ (d) $30°$

30. Three points A, B and C are considered on a parabola. The tangents to the parabola at these points form ΔMNP (NP being tangent at A, PM at B and MN at C). If the line through B and parallel to axis of parabola intersects AC at L, then the quadrilateral $LMNP$

(a) is always a parallelogram
(b) can never be parallelogram
(c) is parallelogram only when ordinates of A, B, C are in AP
(d) has exactly two sides parallel to each other always

Answers

1. (a)	**2.** (d)	**3.** (a)	**4.** (a)	**5.** (c)
6. (a)	**7.** (d)	**8.** (c)	**9.** (c)	**10.** (b)
11. (c)	**12.** (b)	**13.** (d)	**14.** (d)	**15.** (a)
16. (b)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (b)
21. (d)	**22.** (c)	**23.** (c)	**24.** (d)	**25.** (b)
26. (b)	**27.** (b)	**28.** (d)	**29.** (a)	**30.** (a)

ELLIPSE

An ellipse is the locus of a point which moves in a plane in such a way that its distance from a fixed point (i.e. focus) to a fixed line (i.e. directrix) is a constant ratio. This constant ratio is called eccentricity (e), which is less than 1, i.e. $0 < e < 1$.

Equation of an Ellipse in Standard Form

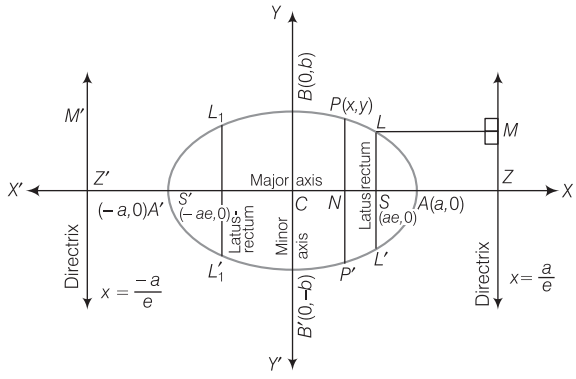
Standard form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$$

where, a and b are some constants and $b^2 = a^2(1 - e^2)$.

Some Important Terms and Properties of an Ellipse

The sketch of the locus of a moving point which satisfy the equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (where $a > b$) is shown in the following figure.



(i) **Symmetry** On replacing y by $(-y)$ and x by $(-x)$, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ remains same. So, the curve is symmetrical about X and Y-axes.

(ii) **Foci** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of an ellipse and the distance between the foci is $SS' = 2ae$.

(iii) **Directrices** The lines ZM and $Z'M'$ are called directrices of the ellipse and its equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$, respectively. Also, the distance between the directrices is $ZZ' = \frac{2a}{e}$.

(iv) **Axes** The lines AA' and BB' are major and minor axes of an ellipse, respectively.
(a) The length of major axis = $AA' = 2a$
(b) The length of minor axis = $BB' = 2b$

(v) **Centre** The point of intersection of the axes of an ellipse is known as the centre (C) of an ellipse.

Note All chords passing through C are bisected at C.

(vi) **Vertices** The end points A and A' of major axis are known as the vertices of an ellipse.
The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

Note The vertex divides internally and externally, the join of focus and point of intersection of directrix with axis in the ratio of e : 1.

(vii) **Focal chord** A chord of an ellipse passing through its focus is called a focal chord.

(viii) **Ordinate and double ordinate** Let P be a point on the ellipse. From P, draw $PN \perp AA'$, then PN is called ordinate and PNP' is called double ordinate of the point P.

(ix) **Latusrectum** The focal chord perpendicular to the major axis is called latusrectum.
If LL' and L_1L_1' are the latusrectum of an ellipse, then these lines are perpendicular to AA' and passing through S and S', respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

$$L_1 \equiv \left(-ae, \frac{b^2}{a}\right), L_1' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

and length of latusrectum = $LL' = \frac{2b^2}{a} = L_1L_1'$

(x) **Focal distance** The distance of any point on the ellipse from the focus is called focal distance.
By definition of an ellipse,

$$\frac{SP}{PM} = e$$

$$\Rightarrow \quad SP = ePM = e\left(\frac{a}{e} - x\right) = a - ex$$

and $$S'P = e\left(\frac{a}{e} + x\right) = a + ex$$

Thus, the focal distances of any point $P(x, y)$ on the ellipse are $(a - ex)$ and $(a + ex)$.

Note $SP + S'P = 2a$, i.e. the sum of the distances of any point $P(x, y)$ on the ellipse from foci is always constant.

(xi) **Eccentricity of an ellipse** Since, $SP = ePM$

$$\therefore \quad (SP)^2 = e^2(PM)^2$$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

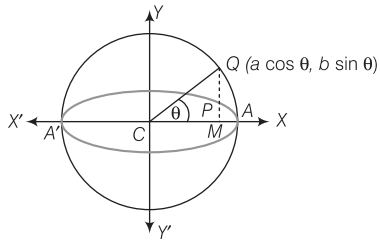
$\Rightarrow \quad (x-ae)^2 + y^2 = (a-xe)^2$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad \ldots(i)$

On comparing Eq. (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get $b^2 = a^2(1-e^2)$ or $e = \sqrt{1 - \frac{b^2}{a^2}}$.

(xii) **Auxiliary circle** A circle drawn by taking major axis (AA') as diameter is called auxiliary circle.

If equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of its auxiliary circle is $x^2 + y^2 = a^2$.



Let Q be any point on the auxiliary circle such that $QM \perp AA'$ cutting the ellipse at point P. The points P and Q are known as corresponding point on the ellipse and auxiliary circle, respectively.

Here, $\angle MCQ = \theta\,(0 \le \theta < 2\pi)$ is known as eccentric angle of any point P on the ellipse.

Note CQ but not CP is inclined at θ with X-axis.

Parametric Equation of an Ellipse

The parametric equations of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $x = a\cos\theta$ and $y = b\sin\theta$, where θ is a parameter and $0 < \theta < 2\pi$.

Equation of a Chord

Equation of chord joining two points P and Q with eccentric angles θ_1 and θ_2, respectively is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

Note (i) Equation of focal chord through $(ae, 0)$ is

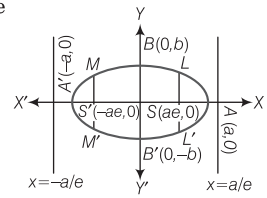
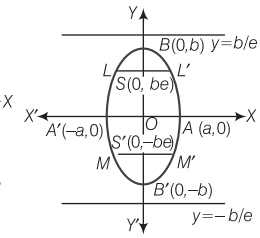
$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) = \frac{e-1}{e+1}.$$

(ii) Equation of focal chord through $(-ae, 0)$ is

$$\tan\left(\frac{\theta_1}{2}\right)\tan\left(\frac{\theta_2}{2}\right) = \frac{e+1}{e-1}.$$

Standard Forms of Ellipse

Two standard forms of the ellipse are shown below alongwith their properties:

Standard equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,(a > b)$ (Horizontal form of an ellipse)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,(a < b)$ (Vertical form of an ellipse)
Shape of the ellipse		
Centre	(0, 0)	(0, 0)
Equation of major axis	$y = 0$	$x = 0$
Equation of minor axis	$x = 0$	$y = 0$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2-b^2}{a^2}}$	$e = \sqrt{\frac{b^2-a^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Ends of latusrectum	$\left(\pm ae, \pm\frac{b^2}{a}\right)$	$\left(\pm\frac{a^2}{b}, \pm be\right)$
Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$
Focal distances of point (x, y)	$SP = a - ex$ and $S'P = a + ex$	$SP = b - ey$ and $S'P = b + ey$
Sum of focal distances $(SP + S'P)$	$2a$	$2b$
Distance between foci	$2ae$	$2be$
Distance between directrices	$\frac{2a}{e}$	$\frac{2b}{e}$
Tangents at the vertices	$x = \pm a$	$y = \pm b$

Various Forms of Equation of Tangent to an Ellipse

1. Point Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Note The equation of tangent at (x_1, y_1) to any second degree curve can also be obtained by replacing x^2 by xx_1, y^2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, xy by $\frac{xy_1+x_1y}{2}$ and without changing the constant (if any) in the equation of curve.

2. Parametric Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at any point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

3. Slope Form

The equation of tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with slope m is $y=mx\pm\sqrt{a^2m^2+b^2}$.

Condition for Tangency and Points of Contact

The line $y=mx+c$ is said to be tangent to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ iff $c^2=a^2m^2+b^2$. The coordinates of point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$.

Various Forms of Equation of Normal to an Ellipse

1. Point Form

The equation of the normal at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$.

2. Parametric Form

The equation of the normal at any point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax\sec\theta-by\operatorname{cosec}\theta)=a^2-b^2$.

3. Slope Form

The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope m is $y=mx\pm\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$.

Condition for Normality and Points of Contact

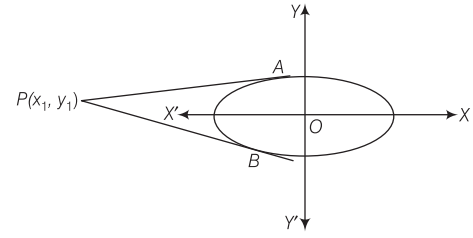
The line $y=mx+c$ is normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ iff $c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$. The coordinates of the point of contact are $\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$.

REMEMBER

- In general, four normals can be drawn to an ellipse from a point in its plane.
- If there are four points on an ellipse at which the normals pass through a common point. These four points are called the conormal points.
- If α, β, γ and δ are the eccentric angles of these four conormal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.
- If α, β and γ are the eccentric angles of these three conormal points, then $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$.

Equation of Pair of Tangents

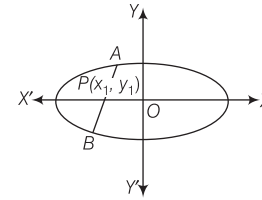
The equation of the pair of tangents drawn from any point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is $SS_1=T^2$



where, $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1, S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1, T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Equation of Chord Bisected at a Given Point

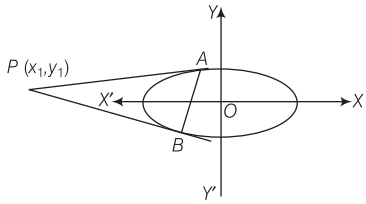
If a point $P(x_1, y_1)$ bisects the chord AB of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of chord AB is given by $T=S_1$



where, $T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1\equiv\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.

Equation of Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $T = 0$



where, $T \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$.

Diameter

The equation of the diameter bisecting chords of slope m of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is given by $y = -\frac{b^2}{a^2 m} x$.

Conjugate Diameters

Two diameters of an ellipse are said to be conjugate diameters, if each bisects the chords parallel to the other. Two straight lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $m_1 m_2 = \frac{-b^2}{a^2}$.

Director Circle

The equation of director circle of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$.

Note The product of perpendiculars from the foci on any tangent to the ellipse is b^2.

Pole and Polar

The polar of a point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

MASTER STROKES

1. The area (in sq units) of the quadrilateral formed by the tangents at the end points of latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is

(a) $\frac{27}{2}$ (b) 27 (c) $\frac{27}{4}$ (d) 18

2. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$, where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $(0, 0)$ and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

(a) 2 (b) 4 (c) 6 (d) 8

3. From a point P, perpendicular tangents PQ and PR are drawn to ellipse $x^2 + 4y^2 = 4$. The locus of the circumcentre of ΔPQR is

(a) $x^2 + y^2 = \frac{16}{5}(x^2 + 4y^2)^2$ (b) $x^2 + y^2 = \frac{5}{16}(x^2 + 4y^2)^2$

(c) $x^2 + 4y^2 = \frac{16}{5}(x^2 + y^2)^2$ (d) $x^2 + 4y^2 = \frac{5}{16}(x^2 + y^2)^2$

4. Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{x^2}{4} = 1$ touching the ellipse at points A and B. The coordinates of A and B are

(a) $(3, 0)$ and $(0, 2)$ (b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $(0, 2)$ (d) $(3, 0)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

5. In Q. 4, the orthocentre of ΔPAB is

(a) $\left(5, \frac{8}{7}\right)$ (b) $\left(\frac{7}{5}, \frac{25}{8}\right)$

(c) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{25}, \frac{7}{5}\right)$

6. In Q. 4, the equation of the locus of the point whose distances from the point P and the line AB are equal, is

(a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(b) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(c) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(d) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

7. The equation of the ellipse whose axes are the coordinate axes which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$, is

(a) $3x^2 + 5y^2 - 32 = 0$ (b) $5x^2 + 3y^2 - 48 = 0$

(c) $3x^2 + 5y^2 - 15 = 0$ (d) $5x^2 + 3y^2 - 32 = 0$

8. The equation of the hyperbola whose foci are $(-2, 0)$, $(2, 0)$ and has eccentricity 2, is given by

(a) $3x^2 - y^2 = 3$ (b) $-x^2 + 3y^2 = 3$

(c) $-3x^2 + y^2 = 3$ (d) $x^2 - 3y^2 = 3$

9. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passes through the point $(0, 4)$ circumscribing the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

10. The tangent at a point $P(\theta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the auxiliary circle at points Q and R. If QR subtends a right angle at the centre C of the ellipse, then the eccentricity of the ellipse is

(a) $\frac{1}{\sqrt{1 + \cos^2\theta}}$ (b) $\frac{1}{\sqrt{1 + \sin^2\theta}}$

(c) $\sqrt{1 + \cos^2\theta}$ (d) $\sqrt{1 + \sin^2\theta}$

11. Let d_1 and d_2 be the lengths of perpendiculars drawn from foci S and S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at any point P on the ellipse. Then, $SP : SP'$ equals

(a) $d_1 : d_2$ (b) $d_2 : d_1$ (c) d_1^2 (d) None of these

12. The eccentricity of an ellipse with centre at the origin and axes along the coordinate axes, is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the equation of the ellipse is

(a) $4x^2 + 3y^2 = 1$ (b) $3x^2 + 4y^2 = 12$

(c) $4x^2 + 3y^2 = 12$ (d) $3x^2 + 4y^2 = 1$

13. If the tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the mid-point of the intercept made by the tangents between the coordinate axes, is

(a) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

(c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

14. If a bar of given length moves with its extremities on two fixed straight lines at right angles, then the locus of any point on the bar describes a/an

(a) circle (b) parabola (c) ellipse (d) hyperbola

15. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points

(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \frac{\sqrt{19}}{4}\right)$

(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

16. If the line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets the auxiliary circle at the point M, then the area of the triangle with vertices of A, M and the origin O, is

(a) $\frac{31}{10}$ sq units (b) $\frac{29}{10}$ sq units

(c) $\frac{21}{10}$ sq units (d) $\frac{27}{10}$ sq units

17. If tangent at a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is drawn which cuts the coordinate axes at A and B, then the minimum area of ΔOAB is (O being origin)

(a) ab (b) $\frac{a^3 + b^3 + ab}{3}$

(c) $a^2 + b^2$ (d) $\frac{a^2 + b^2}{4}$

18. The locus of the foot of the perpendicular from the foci on any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) $x^2 + y^2 = b^2$ (b) $x^2 + y^2 = a^2$

(c) $x^2 + y^2 = a^2 + b^2$ (d) None of these

19. The locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which make complementary angles with X-axis, is

(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 + y^2 = a^2 - b^2$

(c) $x^2 - y^2 = a^2 + b^2$ (d) $x^2 - y^2 = a^2 - b^2$

20. The locus of the foot of the perpendicular drawn from the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on any tangent, is

(a) $(x^2 - y^2)^2 = a^2x^2 + b^2y^2$

(b) $(x^2 - y^2)^2 = a^2x^2 - b^2y^2$

(c) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

(d) $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

21. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latusrectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latusrectum PQ, are

(a) $x^2 + 2\sqrt{3}y = \pm 3\sqrt{3}$ (b) $x^2 \pm 2\sqrt{3}y = 3 \pm \sqrt{3}$

(c) $x^2 + 2\sqrt{3}y = \sqrt{3} \pm 3$ (d) $x^2 - 2\sqrt{3}y = \pm 3\sqrt{3}$

22. The locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles differ by $\frac{\pi}{2}$, is

(a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = b^2$

(c) $x^2 + y^2 = a^2 + b^2$ (d) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

To be Continued at Page 47

TARGET JEE 2017

TEST RIDER

Comprehensive Simulator Test Series for JEE Main & Advanced

JEE MAIN

(PAPER ONE PREP UP)

FULL TEST 3

(COMPLETE SYLLABUS)
FULLY SOLVED

Mock Questions from Complete Syllabus with Complete Solutions

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. If $x>1$, $y>1$, $z>1$ and x, y, z are in GP, then $(\log x^2)^{-1}, (\log xy)^{-1}, (\log xz)^{-1}$ are in
(a) AP *(b)* GP
(c) HP *(d)* None of these

2. If two events A and B are such that $P(\overline{A})=0.3$, $P(B)=0.4$ and $P(A\cap\overline{B})=0.5$, then the value of $P(B/A\cup\overline{B})$ is
(a) 1/2 *(b)* 1/4 *(c)* 3/4 *(d)* 4/5

3. $\int \sin 51x\,(\sin x)^{49}dx$ equals
(a) $\frac{\sin 50x\,(\sin x)^{50}}{50}+C$ *(b)* $\frac{\cos 50x\,(\sin x)^{50}}{50}+C$
(c) $\frac{\cos 50x\,(\cos x)^{50}}{50}+C$ *(d)* $\frac{\sin 50x\,(\sin x)^{51}}{50}+C$

4. If $|z|=1$ and $z\neq\pm1$, then one possible value of $\arg(z)-\arg(z+1)-\arg(z-1)$, is
(a) $\pi/6$ *(b)* $\pi/3$ *(c)* $\pi/2$ *(d)* $\pi/4$

5. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right)=\tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then $\sin x\left(\frac{3+\sin^2 x}{1+3\sin^2 x}\right)$ equals
(a) $\cos y$ *(b)* $\sin y$
(c) $\sin 2y$ *(d)* None of these

6. In ΔABC, if $3\sin A+4\cos B=6$ and $4\sin B+3\cos A=1$, then $\angle C$ equals
(a) $\frac{\pi}{6}$ *(b)* $\frac{\pi}{4}$ *(c)* $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

7. The number of solutions of the equation $\tan x+\sec x=2\cos x$ in the interval $[0, 2\pi]$, is
(a) 0 *(b)* 1
(c) 2 *(d)* 3

8. Let X be a set containing n elements. The number of all ordered triplets (A,B,C) such that C is a subset of B and B is a proper subset of A, where $A\subseteq X$, is
(a) 4^n *(b)* 3^n
(c) 4^n-3^n *(d)* 3^n-2^n

9. If the area bounded by $\frac{x^2}{16}+\frac{y^2}{9}=1$, X-axis and $y=mx$ in Ist quadrant is $\frac{1}{8}$th of area of the ellipse, then the value of m is
(a) $\frac{1}{2}$ *(b)* $\frac{2}{3}$ *(c)* $\frac{3}{4}$ *(d)* $\frac{4}{5}$

10. $\lim_{x\to0} x^2\left(1+2+3+\ldots+\left[\frac{1}{|x|}\right]\right)$, where $[\cdot]$ denotes greatest integer function, is equal to
(a) 1 *(b)* $\frac{3}{2}$ *(c)* $\frac{1}{2}$ *(d)* 2

11. If the solution of $x^3\frac{dy}{dx}+4x^2\tan y=e^x\sec y$ satisfying $y(1)=0$ is $\sin y=e^x(x-1)x^{-k}$, then the value of k is
(a) 1 *(b)* 2
(c) 3 *(d)* 4

12. The equation of plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$, is

(a) $x+2y-2z=0$ (b) $3x+2y-2z=0$
(c) $x-2y+z=0$ (d) $5x+2y-4z=0$

13. Let α, β and γ be the roots of the equation $x^3+3x^2-6x-8=0$. If $\left(\frac{1}{\alpha},\alpha\right), \left(\frac{1}{\beta},\beta\right)$ and $\left(\frac{1}{\gamma},\gamma\right)$ are the vertices of a triangle, then

(a) centroid of a triangle is $\left(\frac{-3}{4},-1\right)$

(b) orthocentre of a triangle is $\left(\frac{-1}{8},-8\right)$

(c) centroid of a triangle is $\left(\frac{1}{4},1\right)$

(d) orthocentre of a triangle is $\left(-8,\frac{-1}{8}\right)$

14. Let $f:R\to R$ be a positive decreasing function with $\lim\limits_{x\to\infty}\frac{f\left(\frac{x-x^3}{6}\right)}{f(x)}=1$, then $\lim\limits_{x\to\infty}\frac{f(\sin x)}{f(x)}$ is equal to

(a) 1 (b) $\pi/2$
(c) $1/6$ (d) $\pi/36$

15. If the eccentricity of the hyperbola $x^2-y^2\sec^2\alpha=5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha+y^2=25$, then the value of α is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

16. Line $x+y=n\,(n\in N)$ cuts the coordinate axes at A and B. If O is origin, then number of points, having both coordinate as integer that lie in ΔAOB, is

(a) $\frac{n^2-4n+3}{2}$ (b) $\frac{n^2+n}{2}$
(c) $\frac{n^2-n}{2}$ (d) $\frac{n^2-3n+2}{2}$

17. If tangents are drawn from (4, 4) to the circle $x^2+y^2-2x-2y-7=0$ to meet the circle at A and B, then the length of the chord AB is

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$
(c) $2\sqrt{6}$ (d) $6\sqrt{2}$

18. If $f'(1)=1, \frac{df(2x)}{dx}=f'(x), \forall\ x>0$ and $f'(x)$ is differentiable, then there exists a number $c\in(2,4)$ such that $f''(c)$ is equal to

(a) $-\frac{1}{4}$ (b) $-\frac{1}{8}$
(c) $\frac{1}{4}$ (d) $\frac{1}{8}$

19. If A is an orthogonal matrix of order n, then the value of $|\text{adj}\cdot(\text{adj}\,A)|$ is

(a) 0 (b) ±1 (c) n (d) $n-2$

20. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be unit vectors and $\mathbf{c}$ is a vector such that $\hat{\mathbf{a}}\times\hat{\mathbf{b}}+\hat{\mathbf{a}}=\mathbf{c}$ and $\mathbf{c}\times\hat{\mathbf{a}}=\hat{\mathbf{b}}$, then the value of $[\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \mathbf{c}]$ is

(a) 0 (b) -1
(c) 1 (d) None of these

21. The distance between the plane $x-2y+z=6$ and the plane containing the set of points $(1+2\lambda, 2+3\lambda, 3+4\lambda)$ and $(2+3\mu, 3+4\mu, 4+5\mu)$, where λ and μ are parameters, is

(a) $\sqrt{3/2}$ (b) $\sqrt{6}$ (c) $\sqrt{12}$ (d) $2\sqrt{6}$

22. The number of ways in which 5 A's and 6 B's can be arranged in a row which reads the same backwards and forwards, is

(a) 8 (b) 9
(c) 10 (d) 11

23. The coefficient of x^{50} in the expansion $(1+x)^{100}+2x(1+x)^{99}+3x^2\ (1+x)^{98}+\ldots+101x^{100}$ is

(a) ${}^{100}C_{50}$ (b) ${}^{101}C_{51}$ (c) ${}^{102}C_{50}$ (d) ${}^{100}C_{51}$

24. If a determinant of order 3×3 is formed by using the numbers 1 or -1, then the minimum value of the determinant is

(a) -2 (b) -4 (c) 0 (d) -8

25. Two circles with centres A and B, touch at T, BD is the tangent at D and TC is common tangent. AT has length 3 and BT has length 2. The length of CD is

(a) $\frac{4}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) $\frac{7}{4}$

26. Let $f(x)=\int_2^x\frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f, then the value of $g'(0)$ is

(a) 1 (b) $\sqrt{17}$
(c) 17 (d) None of these

27. If the vertex of a parabola is at (3, 2) and its directrix is the line $x-y+1=0$, then the equation of its latusrectum is

(a) $x-y=2$ (b) $x-y=3$
(c) $x-y=1$ (d) $x+y=2$

28. The negation of $p\to(\sim p\vee q)$ is

(a) $p\vee(p\vee\sim q)$ (b) $p\to\sim(p\vee q)$
(c) $p\to q$ (d) $p\wedge\sim q$

29. If the variables take the values 0, 1, 2, 3, ..., 10 with frequencies ${}^{10}C_0, {}^{10}C_1, {}^{10}C_2, \ldots, {}^{10}C_{10}$ respectively, then the variables of the distribution is

(a) 10 (b) $\sqrt{10}$
(c) 5 (d) 5/2

30. If the roots of the equation $x^3-ax^2+14x-8=0$ are all real and positive, then the minimum value of a is

(a) 6 (b) 5 (c) 4 (d) 8

Answers with Explanation

1. *(c)* We have, $x>1$, $y>1$, $z>1$ and x, y, z are in GP.

$\therefore \quad y^2 = xz$

Taking log on both sides, we get

$\log y^2 = \log xz \Rightarrow 2\log y = \log x + \log z$

Hence, $\log x$, $\log y$, $\log z$ are in AP.

On adding $\log x$, we get

$2\log x$, $\log xy$, $\log xz$ are in AP.

$\therefore$ $\log x^2$, $\log xy$, $\log xz$ are in AP.

Hence, $\frac{1}{\log x^2}, \frac{1}{\log xy}, \frac{1}{\log xz}$ are in HP.

2. *(b)* We have, $P(\overline{A}) = 0.3$, $P(B) = 0.4$, $P(A \cap \overline{B}) = 0.5$

Now, $P(B/A \cup \overline{B}) = \frac{P(B \cap (A \cup \overline{B}))}{P(A \cup \overline{B})}$

$= \frac{P(A) - P(A \cap \overline{B})}{P(A) + P(\overline{B}) - P(A \cap \overline{B})}$

$= \frac{0.7 - 0.5}{0.7 + 0.6 - 0.5} = \frac{0.2}{0.8} = \frac{1}{4}$

3. *(a)* Let $I = \int \sin 51x (\sin x)^{49} dx$

$\Rightarrow I = \int \sin(50x + x)(\sin x)^{49} dx$

$\Rightarrow I = \int (\sin 50x \cos x + \cos 50x \sin x)(\sin x)^{49} dx$

$\Rightarrow I = \int [\sin 50x \cos x (\sin x)^{49} + \cos 50x (\sin x)^{50}] dx$

$\Rightarrow I = \frac{1}{50} \int [50 \sin 50x \cos x (\sin x)^{49} + 50 \cos 50x (\sin x)^{50}] dx$

$\Rightarrow I = \frac{1}{50} \int \frac{d}{dx} \sin 50x (\sin x)^{50}$

$\therefore I = \frac{\sin 50x (\sin x)^{50}}{50} + C$

4. *(c)* Given, $|z| = 1$ and $z \neq \pm 1$

$\therefore \arg(z) - \arg(z+1) - \arg(z-1)$

$= \arg\left(\frac{z}{z^2 - 1}\right) = \arg\left(\frac{z}{z^2 - z\bar{z}}\right) \quad [\because |z|^2 = z\bar{z} = 1]$

$= \arg\left(\frac{z}{z(z - \bar{z})}\right) = \arg\left(\frac{1}{z - \bar{z}}\right)$

$= \arg$ (purely imaginary number) $[\because z - \bar{z} = 2\,\text{Im}(z)]$

$= \frac{\pi}{2}$

5. *(b)* Given, $\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$\Rightarrow \frac{1 + \tan\frac{y}{2}}{1 - \tan\frac{y}{2}} = \left(\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right)^3$

$\Rightarrow \frac{\cos\frac{y}{2} + \sin\frac{y}{2}}{\cos\frac{y}{2} - \sin\frac{y}{2}} = \left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right)^3$

On squaring both sides, we get

$\frac{1 + \sin y}{1 - \sin y} = \left(\frac{1 + \sin x}{1 - \sin x}\right)^3$

Using componendo and dividendo, we get

$\frac{1 + \sin y + 1 - \sin y}{1 + \sin y - 1 + \sin y} = \frac{(1 + \sin x)^3 + (1 - \sin x)^3}{(1 + \sin x)^3 - (1 - \sin x)^3}$

$\Rightarrow \frac{2}{2\sin y} = \frac{2(1 + 3\sin^2 x)}{2\sin x (3 + \sin^2 x)}$

$\Rightarrow \sin x \left(\frac{3 + \sin^2 x}{1 + 3\sin^2 x}\right) = \sin y$

6. *(a)* Given, $3\sin A + 4\cos B = 6$ …(i)

and $4\sin B + 3\cos A = 1$ …(ii)

On squaring and adding Eqs. (i) and (ii), we get

$24(\sin A \cos B + \cos A \sin B) = 12$

$\Rightarrow \sin(A + B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2} \Rightarrow C = \frac{\pi}{6}$

7. *(c)* We have, $\tan x + \sec x = 2\cos x$

$\Rightarrow 1 + \sin x = 2\cos^2 x$

$\Rightarrow 1 + \sin x = 2 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow (\sin x + 1)(2\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}, -1$

$\Rightarrow \sin x \neq -1, \cos x \neq 0$

Hence, it has two solutions in the interval $[0, 2\pi]$.

8. *(c)* We have, C is a subset of B, B is a proper subset of A and $A \subseteq X$.

$\therefore \quad C \subseteq B \subset A \subseteq X$

Therefore, total number of ways

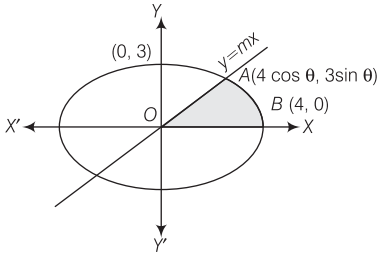
$= \sum_{i=0}^{n} {}^nC_i \left[\sum_{j=0}^{i-1} {}^iC_j \left(\sum_{k=0}^{j} {}^jC_k\right)\right]$

$= \sum_{i=0}^{n} {}^nC_i \left[\sum_{j=0}^{i-1} {}^iC_j 2^j\right]$

$= \sum_{i=0}^{n} {}^nC_i [(1+2)^i - 2^i] = \sum_{i=0}^{n} {}^nC_i (3^i - 2^i)$

$= \sum_{i=0}^{n} {}^nC_i 3^i - \sum_{i=0}^{n} {}^nC_i 2^i = (1+3)^n - (1+2)^n = 4^n - 3^n$

9. *(c)* Given, equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.



We know that, area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq units

and area of shaded region $OAB=\frac{ab}{2}\theta$ sq unit

[$\because \theta$ is an eccentric angle]

Since, $\frac{1}{8}\times$ Area of ellipse = Area of shaded region OAB

$\therefore \quad \frac{\pi ab}{8}=\frac{ab}{2}\theta \Rightarrow \theta=\frac{\pi}{4}$

$\therefore$ Slope of line $OA\ m=\frac{3}{4}\tan\theta \Rightarrow m=\frac{3}{4}$

10. ***(c)*** We have, $\lim\limits_{x\to 0}x^2\left(1+2+3+\ldots+\left[\frac{1}{|x|}\right]\right)$

$$=\lim_{x\to 0}\frac{x^2\left[\frac{1}{|x|}\right]\left(1+\left[\frac{1}{|x|}\right]\right)}{2}$$

$$\left[\because 1+2+3+\ldots+n=\frac{n(n+1)}{2}\right]$$

Now, using the property that

$$\frac{1}{|x|}-1<\left[\frac{1}{|x|}\right]\le\frac{1}{|x|}, \text{ we get}$$

$$\frac{1}{2}|x|<\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)\left[\frac{1}{|x|}\right]}{2}\le\frac{1}{2}(1+|x|)$$

Now, applying Sandwich theorem, we get the required limit is 1/2.

11. ***(d)*** We have, $x^3\frac{dy}{dx}+4x^2\tan y=e^x\sec y$

$$\Rightarrow \quad \cos y\frac{dy}{dx}+\frac{4}{x}\sin y=\frac{e^x}{x^3}$$

$$\Rightarrow \quad \frac{dz}{dx}+\frac{4z}{x}=\frac{e^x}{x^3} \quad \left[\text{put } z=\sin y \Rightarrow \frac{dz}{dx}=\cos y\frac{dy}{dx}\right]$$

IF $=e^{\int\frac{4}{x}dx}=x^4$

$$\therefore \quad zx^4=\int xe^x dx$$

$$\Rightarrow \quad zx^4=e^x(x-1)+C$$

$$\Rightarrow \quad x^4\sin y=e^x(x-1)+C$$

When $x=1,\ y=0 \Rightarrow C=0$

$$\therefore \quad x^4\sin y=e^x(x-1)$$

$$\Rightarrow \quad \sin y=e^x(x-1)x^{-4}$$

$$\therefore \quad k=4$$

12. ***(c)*** Vector along the required plane is

$$\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & 4 & 2\\ 4 & 2 & 3\end{vmatrix}=8\hat{\mathbf{i}}-\hat{\mathbf{j}}-10\hat{\mathbf{k}}$$

So, normal vector ($\mathbf{n}$) to the plane is

$$\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 8 & -1 & -10\\ 2 & 3 & 4\end{vmatrix}=26\hat{\mathbf{i}}-52\hat{\mathbf{j}}+26\hat{\mathbf{k}}$$

So, equation of the plane is $\mathbf{r}\cdot\mathbf{n}=0$.

$$\therefore \quad (x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})\cdot(26\hat{\mathbf{i}}-52\hat{\mathbf{j}}+26\hat{\mathbf{k}})=0$$

$$\Rightarrow \quad 26x-52y+26z=0$$

$$\Rightarrow \quad x-2y+z=0$$

13. ***(d)*** We have, $x^3+3x^2-6x-8=0$

Since, α, β, γ are the roots of the equation.

$\therefore \quad \Sigma\alpha=-3, \Sigma\alpha\beta=-6, \Sigma\alpha\beta\gamma=8$

Here, $\left(\frac{1}{\alpha},\alpha\right),\left(\frac{1}{\beta},\beta\right)$ and $\left(\frac{1}{\gamma},\gamma\right)$ are the vertices of triangle.

$\therefore$ Coordinates of the centroid of triangle

$$=\left[\frac{1}{3}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right),\frac{1}{3}(\alpha+\beta+\gamma)\right]$$

$$=\left(\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{3\alpha\beta\gamma},\frac{\alpha+\beta+\gamma}{3}\right)$$

$$=\left(\frac{-6}{3\times 8},\frac{-3}{3}\right)=\left(\frac{-1}{4},-1\right)$$

Since, $\left(\frac{1}{\alpha},\alpha\right),\left(\frac{1}{\beta},\beta\right)$ and $\left(\frac{1}{\gamma},\gamma\right)$ lie on hyperbola $xy=1$.

Therefore, orthocentre of triangle will be $\left(-\alpha\beta\gamma,\frac{-1}{\alpha\beta\gamma}\right)$

$$=\left(-8,\frac{-1}{8}\right)$$

14. ***(a)*** We have, $\lim\limits_{x\to\infty}\frac{f\left(\frac{x-x^3}{6}\right)}{f(x)}=1$

Since, $f: R\to R$ is decreasing function.

$$\therefore \quad \frac{x-x^3}{6}<\sin x<x, \forall\ x>0$$

$$\Rightarrow \quad f\left(\frac{x-x^3}{6}\right)>f(\sin x)>f(x)$$

$$\Rightarrow \quad \frac{f\left(\frac{x-x^3}{6}\right)}{f(x)}>\frac{f(\sin x)}{f(x)}>1$$

$\therefore$ By Sandwich theorem, $\lim\limits_{x\to\infty}\frac{f(\sin x)}{f(x)}=1$

15. ***(b)*** Given, equation of hyperbola is

$$x^2-y^2\sec^2\alpha=5 \quad \text{or} \quad \frac{x^2}{5}-\frac{y^2}{5\cos^2\alpha}=1$$

$\therefore$ Eccentricity of hyperbola,

$$e_1=\sqrt{1+\frac{5}{5\cos^2\alpha}}=\sqrt{\frac{1+\cos^2\alpha}{\cos^2\alpha}}$$

and equation of ellipse is $x^2\sec^2\alpha+y^2=25$

or $$\frac{x^2}{25\cos^2\alpha}+\frac{y^2}{25}=1$$

Eccentricity of ellipse,

$$e_2=\sqrt{1-\frac{25}{25\cos^2\alpha}}=\sqrt{\frac{1-\cos^2\alpha}{\cos^2\alpha}}$$

According to the question, $e_1=\sqrt{3}e_2$

$$\Rightarrow \quad \sqrt{\frac{1+\cos^2\alpha}{\cos^2\alpha}}=\sqrt{3}\sqrt{\frac{1-\cos^2\alpha}{\cos^2\alpha}}$$

$$\Rightarrow \quad 1+\cos^2\alpha=3-3\cos^2\alpha$$

$$\Rightarrow \quad \cos\alpha=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \alpha=\frac{\pi}{4}$$

16. ***(d)*** Let (α, β) be the point in ΔAOB.

$\therefore \quad \alpha + \beta < n \quad \Rightarrow \quad \alpha + \beta \le n - 1$

$\Rightarrow \quad 1 \le \alpha \le n - 1, \quad 1 \le \beta \le n - 1, \quad \alpha, \beta \in N$

$\therefore$ Number of points (α, β)

$$= {}^{n-1-2+2}C_2 = {}^{n-1}C_2$$

$$= \frac{(n-1)(n-2)}{2} = \frac{n^2 - 3n + 2}{2}$$

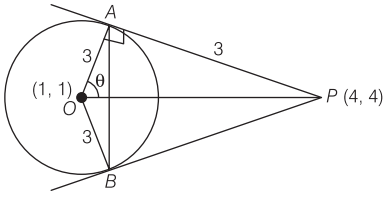
17. ***(b)*** Given, equation of circle is

$$x^2 + y^2 - 2x - 2y - 7 = 0$$

Length of tangent PA from point $P(4, 4)$

$$= \sqrt{(4)^2 + (4)^2 - 2(4) - 2(4) - 7} = 3$$

and radius of circle $= \sqrt{(1)^2 + (1)^2 + 7} = 3$



In ΔOAP, $\quad \tan\theta = \frac{PA}{OA} = \frac{3}{3} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \quad \angle AOB = 2\angle AOP = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$

In ΔAOB, $\quad \angle O = 90°$

$\therefore \quad AB^2 = OA^2 + OB^2 = 3^2 + 3^2 = 18$

$\Rightarrow \quad AB = \sqrt{18} = 3\sqrt{2}$

18. ***(b)*** We have, $\quad f'(1) = 1$ and $\frac{df(2x)}{dx} = f'(x)$

$\therefore \quad \frac{df(2x)}{dx} = 2f'(2x) = f'(x)$

$\Rightarrow \quad f'(2x) = \frac{1}{2}f'(x) \Rightarrow f'(2) = \frac{1}{2}f'(1)$

$\Rightarrow \quad f'(2) = \frac{1}{2}$ and $f'(4) = \frac{1}{2}f'(2)$

$\Rightarrow \quad f'(4) = \frac{1}{4}$

By LMVT, $f''(c) = \frac{f'(4) - f'(2)}{4 - 2}$

$\Rightarrow \quad f''(c) = \frac{\frac{1}{4} - \frac{1}{2}}{2} = -\frac{1}{8}$

19. ***(b)*** Since, A is an orthogonal matrix.

$\therefore \quad AA' = I$

$\Rightarrow \quad |AA'| = |I|$

$\Rightarrow \quad |A||A'| = 1$

$\Rightarrow \quad |A|^2 = 1$

$\Rightarrow \quad |A| = \pm 1$

$|\text{adj} \cdot (\text{adj } A)| = |A|^{(n-1)^2} = (\pm 1)^{(n-1)^2} = \pm 1$

20. ***(c)*** Given, $\quad (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) + \hat{\mathbf{a}} = \mathbf{c}$ and $\mathbf{c} \times \hat{\mathbf{a}} = \hat{\mathbf{b}}$

$\Rightarrow \quad (\hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{a}}) \times \hat{\mathbf{a}} = \mathbf{c} \times \hat{\mathbf{a}}$

$\Rightarrow \quad (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \times \hat{\mathbf{a}} = \hat{\mathbf{b}}$

$\Rightarrow \quad (\hat{\mathbf{a}} \cdot \hat{\mathbf{a}})\hat{\mathbf{b}} - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{a}} = \hat{\mathbf{b}}$

$\Rightarrow \quad (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{a}} = 0 \qquad [\because |\hat{\mathbf{a}}| = 1$

$\Rightarrow \quad \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = 0$

Now, $\quad [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \mathbf{c}] = \hat{\mathbf{a}} \cdot (\hat{\mathbf{b}} \times \mathbf{c})$

$= \hat{\mathbf{a}} \cdot [\hat{\mathbf{b}} \times (\hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{a}})]$

$= \hat{\mathbf{a}} \cdot [\hat{\mathbf{b}} \times (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) + \hat{\mathbf{b}} \times \hat{\mathbf{a}}]$

$= \hat{\mathbf{a}} \cdot [(\hat{\mathbf{b}} \cdot \hat{\mathbf{b}})\ \hat{\mathbf{a}} - (\hat{\mathbf{b}} \cdot \hat{\mathbf{a}})\ \hat{\mathbf{b}}] + \hat{\mathbf{b}} \times \hat{\mathbf{a}}]$

$= \hat{\mathbf{a}} \cdot [[\hat{\mathbf{a}} - 0] + \hat{\mathbf{b}} \times \hat{\mathbf{a}}]$

$= \hat{\mathbf{a}} \cdot \hat{\mathbf{a}} + \hat{\mathbf{a}} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{a}})$

$= 1 + [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{a}}] = 1$

Hence, $\quad [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}] = 1$

21. ***(b)*** Normal vector of the plane $= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= (15 - 16)\hat{\mathbf{i}} - (10 - 12)\hat{\mathbf{j}} + (8 - 9)\hat{\mathbf{k}}$

$= -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Equation of plane passing through (1, 2, 3) and normal vector $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ is

$$-(x - 1) + 2(y - 2) - 1(z - 3) = 0$$

$\Rightarrow \quad x - 2y + z = 0$

Since, $x - 2y + z = 6$ and $x - 2y + z = 0$ and parallel planes.

$\therefore$ Distance between the plane is

$$D = \left| \frac{6 - 0}{\sqrt{1^2 + 2^2 + 1^2}} \right| = \left| \frac{6}{\sqrt{6}} \right| = \sqrt{6}$$

22. ***(c)*** We have, 5A's and 6B's

i.e. $\quad AAAAA, BBBBBB$

Since, word reads the same backwards and forwards.

Thus, the middle digits must be A so that even number of A's and B's are available for arrangement.

i.e. $\quad \times\times\times\times\times A \times\times\times\times\times$

Take $AABBB$ on one side of A (middle digit) and their image about middle digit (A) in a unique way.

$\therefore$ Number of arrangements of $AABBB = \frac{5!}{2!\,3!} = 10$

23. ***(c)*** Let $S = (1 + x)^{100} + 2x(1 + x)^{99} + 3x^2(1 + x)^{98} + \ldots + 101x^{100}$ ...(i)

On multiplying both sides by $\frac{x}{1 + x}$, we get

$$\frac{x}{1 + x} S = x(1 + x)^{99} + 2x^2(1 + x)^{98} + 3x^3(1 + x)^{97} + \ldots + \frac{101x^{101}}{(1 + x)} \quad \ldots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$S\left(1 - \frac{x}{1 + x}\right) = (1 + x)^{100} + x(1 + x)^{99} + x^2(1 + x)^{98} + \ldots - \frac{101x^{101}}{(1 + x)}$$

$$\Rightarrow \quad \frac{S}{1 + x} = \frac{(1 + x)^{100}\left[1 - \left(\frac{x}{1 + x}\right)^{101}\right]}{\left(1 - \frac{x}{1 + x}\right)} - \frac{101x^{101}}{(1 + x)}$$

or $\quad S = (1 + x)^{102} - x^{101}(1 + x) - 101x^{101}$

$\therefore$ Coefficient of x^{50} in the expansion is equal to coefficient of x^{50} in the expansion of $(1+x)^{102}$, which is equal to ${}^{102}C_{50}$.

24. ***(b)*** Let $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Applying $C_2 \to C_2 - \frac{a_{12}}{a_{11}} C_1$ and $C_3 \to C_3 - \frac{a_{13}}{a_{11}} C_1$, we get

$$D = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} \times a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

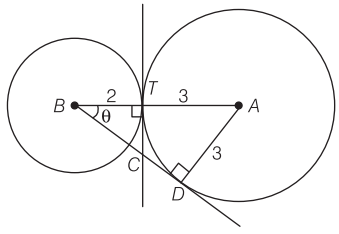
Since, each element of determinant is 1 or –1.

$\therefore$ The minimum value of D is – 4.

25. ***(b)*** In ΔABD, $\quad BD^2 = AB^2 - AD^2$

$\Rightarrow \quad BD^2 = 25 - 9 = 16$

$\Rightarrow \quad BD = 4$



In ΔABD,

$$\cos\theta = \frac{BD}{AB} = \frac{4}{5}$$

In ΔBCT,

$$\cos\theta = \frac{BT}{BC}$$

$$\Rightarrow \quad \frac{4}{5} = \frac{2}{BC}$$

$$\Rightarrow \quad BC = \frac{5}{2}$$

$$\therefore \quad CD = BD - BC = 4 - \frac{5}{2} = \frac{3}{2}$$

26. ***(b)*** We have, $\quad f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$

$$\Rightarrow \quad f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx} \qquad [\because f(x) = y]$$

Since, g is the inverse of f.

$$\therefore \quad g'(y) = \frac{dx}{dy} = \sqrt{1+x^4}$$

$$\therefore \quad y = \int_2^x \frac{dt}{\sqrt{1+t^4}}$$

When $y = 0$, then $x = 2$

$$\therefore \quad g'(y) = \sqrt{1+x^4}$$

$$\Rightarrow \quad g'(0) = \sqrt{1+2^4}$$

$$= \sqrt{1+16} = \sqrt{17}$$

27. ***(b)*** Given, equation of directrix is $x - y + 1 = 0$. …(i)

Equation of axis of parabola is perpendicular to directrix.

$\therefore$ Equation of axis of parabola is

$$x + y = \lambda$$

Since, it passes through vertex (3, 2).

$$\therefore \quad 3 + 2 = \lambda \Rightarrow \lambda = 5$$

$\therefore$ Equation of axis of parabola is

$$x + y = 5 \qquad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get $x = 2, y = 3$

Vertex of parabola is mid-point of focus and coordinate of directrix.

$$\therefore \quad 3 = \frac{x_1 + 2}{2},\ 2 = \frac{y_1 + 3}{2}$$

$$\Rightarrow \quad x_1 = 4,\ y_1 = 1$$

$\therefore$ Focus of parabola is (4, 1).

Now, equation of latusrectum is parallel to the directrix.

$$\therefore \quad x - y = k$$

Since, latusrectum passes through (4, 1).

$$\therefore \quad k = 3$$

Hence, equation of latusrectum is $x - y = 3$.

28. ***(d)*** The negation of $p \to (\sim p \vee q)$

i.e. $\quad \sim(p \to (\sim p \vee q)$

$= p \wedge \sim(\sim p \vee q) \quad [\because \sim(p \to q) = p \wedge \sim q)]$

$= p \wedge (p \wedge \sim q) \quad [\because \sim(p \vee q) = \sim p \wedge \sim q]$

$= p \wedge \sim q$

29. ***(d)*** $\bar{x} = \frac{\sum_{r=0}^{10} r \cdot {}^{10}C_r}{\sum_{r=0}^{10} {}^{10}C_r} = \frac{10 \cdot 2^9}{2^{10}} = 5 \quad \left[\because \sum_{r=0}^{n} r \cdot {}^nC_r = n \cdot 2^{n-1}\right]$

Also, $\quad \sigma^2 = \frac{\sum_{r=0}^{10} r^2 \cdot {}^{10}C_r}{\sum_{r=0}^{10} {}^{10}C_r} - (\bar{x})^2$

$$= \frac{10(10+1)2^{10-2}}{2^{10}} - (5)^2$$

$$\left[\because \sum_{r=0}^{n} r^2 \cdot {}^nC_r = n(n+1)2^{n-2}\right]$$

$$= \frac{10 \times 11 \times 2^8}{2^{10}} - 25$$

$$= \frac{55}{2} - 25 = \frac{5}{2}$$

30. ***(a)*** We have, $\quad f(x) = x^3 - ax^2 + 14x - 8 = 0$

Let α, β, γ be the roots of the equation.

where, $\quad \alpha, \beta, \gamma > 0$

Then, $\quad \alpha + \beta + \gamma = a$

and $\quad \alpha\beta\gamma = 8$

We know that, $\quad AM \geq GM$

$$\therefore \quad \frac{\alpha + \beta + \gamma}{3} \geq (\alpha\beta\gamma)^{1/3}$$

$$\Rightarrow \quad \frac{a}{3} \geq (8)^{1/3}$$

$$\Rightarrow \quad \frac{a}{3} \geq 2$$

$$\Rightarrow \quad a \geq 6$$

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JEE MAIN

(PAPER TWO SCALE UP)

FULL TEST 4

(COMPLETE SYLLABUS) WITH ANSWERS

Mock Questions from complete syllabus to enhance Your Problem Solving Skills

INSTRUCTIONS

- This test consists of 30 questions and each question is allotted 4 marks for correct response.
- Candidates will be awarded marks as stated above for correct response of each question. 1/4 mark will be deducted for indicating incorrect response of each question. No deduction from the total score will be made if no response is indicated for an item in the answer sheet.
- There is only one correct response for each question. Filling up more than one response in any question will be treated as wrong response and marks for wrong response will be deducted according as per instructions.

1. The slope of the tangent to curve $y = f(x)$ at $\{x, f(x)\}$ is $2x + 1$. If the curve passes through the point $(1, 2)$, then the area bounded by the curve, the X-axis and the line $x = 1$, is

(a) 6 sq units *(b)* 5 sq units *(c)* $\frac{5}{6}$ sq unit *(d)* 7 sq units

2. If 6 parallel lines are intersected by 3 parallel lines, then number of parallelograms, thus formed, is

(a) 45 *(b)* 50 *(c)* 65 *(d)* 55

3. If a stick of length 5 units with its ends on two mutually perpendicular lines, then the locus of the middle point of the stick is

(a) $4x^2 + 4y^2 = 5$ *(b)* $4x^2 + 4y^2 = 25$
(c) $x^2 + y^2 = 25$ *(d)* $x^2 + y^2 = 5$

4. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the value of k is

(a) $\frac{1}{4}$ *(b)* 4 *(c)* $\frac{1}{8}$ *(d)* $\frac{1}{4}$

5. Let P be a variable point on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ with foci F_1 and F_2. If A is the area of ΔPF_1F_2, then maximum value of A is

(a) 10 *(b)* 22 *(c)* 15 *(d)* 12

6. $\int_{-\pi/4}^{\pi/4} \frac{\theta + \frac{\pi}{4}}{2 - \cos 2\theta}\, d\theta$ is equal to

(a) $\frac{\pi^2}{3\sqrt{3}}$ *(b)* $3\sqrt{3}\pi^2$ *(c)* $\frac{1}{\sqrt{3}}\pi^2$ *(d)* $\frac{\pi^2}{6\sqrt{3}}$

7. If $\lim_{x \to 0} \frac{\log(5 + 2x) - \log(5 - 2x)}{2x} = k$, then the value of k is

(a) $\frac{2}{5}$ *(b)* $\frac{3}{5}$ *(c)* 5 *(d)* $\frac{1}{5}$

8. The number of terms in $(1 + x)^{201}(x^2 - x + 1)^{200}$ is

(a) 402 *(b)* 202
(c) 502 *(d)* 102

9. For two data sets, each of size 5, the variance are given to be 4, 5 and the corresponding means are given to be 2, 4 respectively. The variance of the combined data set is

(a) 6 *(b)* $\frac{13}{2}$ *(c)* $\frac{11}{2}$ *(d)* $\frac{5}{2}$

10. The angle of elevation of the top of a hill from a point is 30°. After walking 500 m towards the top up a slope inclined at an angle 15° to the horizon, the angle of elevation of the top becomes 60°. Then, the height of the hill is

(a) 500 m *(b)* $500\sqrt{2}$ m
(c) 250 m *(d)* $250\sqrt{2}$ m

11. Three persons entered in the elevator on the ground floor of a tenth floor building. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first. Probability of all three persons leaving at different floors is

(a) $\frac{56}{81}$ *(b)* $\frac{46}{81}$ *(c)* $\frac{3}{10}$ *(d)* $\frac{1}{3}$

12. If $x = \cot^{-1}[\sqrt{\cos 2\theta}] - \tan^{-1}[\sqrt{\cos 2\theta}]$, then $\sin x$ is equal to
(a) $\cos^2 x$ (b) $\tan^2 x$
(c) $\cot^2 x$ (d) None of these

13. If $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then $(\mathbf{a} \times \hat{\mathbf{i}}) \cdot (\mathbf{a} \times \hat{\mathbf{j}}) + 2xy$ equals
(a) 0 (b) xy
(c) 1 (d) 2

14. The nearest point on the straight line $2x - y + 11 = 0$ from the circle $x^2 + y^2 + 2x - \frac{1}{2}y - \frac{25}{8} = 0$, is
(a) $\left(2, \frac{-9}{2}\right)$ (b) $\left(2, \frac{9}{2}\right)$
(c) $\left(\frac{-9}{2}, 2\right)$ (d) $\left(\frac{9}{2}, 2\right)$

15. If $\sin\theta$ and $\cos\theta$ are the roots of the equation $px^2 + qx + r = 0$, then the value of $p^2 - q^2 + 2pr$ is
(a) 0 (b) 1
(c) 2 (d) None of these

16. The number of discontinuities of the greatest integer function $f(x) = [x]$; $x \in \left(-\frac{5}{2}, 200\right)$, is
(a) 200 (b) 205
(c) 201 (d) 202

17. The angle between the lines $x = 2$, $y = 4$ and $y = -2$, $z = 0$ is
(a) 60° (b) 90°
(c) 0° (d) 30°

18. $\int e^{\sqrt{\tan x}} \sec^2 x \sqrt{\cot x}\, dx$ equals
(a) $2e^{\sqrt{\sin x}} + C$ (b) $2e^{\sqrt{\tan x}} + C$
(c) $\frac{1}{2}e^{\sqrt{\sin x}} + C$ (d) $\frac{1}{2}e^{\sqrt{\tan x}} + C$

19. The set of points where the function $f(x) = |x - 5| \log x$ is differentiable, is
(a) $R - \{5\}$ (b) R
(c) $R - \{0\}$ (d) $R - \{-5\}$

20. The absolute maximum value for the function $f(x) = \frac{1}{|x-4|+1} + \frac{1}{|x+8|+1}$ is
(a) $\frac{14}{13}$ (b) $\frac{13}{14}$
(c) 4 (d) 8

21. If $a > 0$, then $\sum_{n=1}^{\infty}\left(\frac{2a}{1+2a}\right)$ is equal to
(a) $2a$ (b) a
(c) $4a$ (d) None of these

22. If the complex number z lies on a circle with centre at the origin and radius $\frac{1}{4}$, then the complex number $-2 + 8z$ lies on a circle with radius
(a) 4 (b) 3 (c) 2 (d) 1

23. The sum of the series $1 + \frac{4}{3} + \frac{10}{9} + \frac{28}{27} + \ldots$ to n terms, is
(a) $\frac{5}{6}n + \frac{1}{6} - \frac{1}{3\cdot 2^n - 1}$ (b) $n + \frac{1}{2} - \frac{1}{2\cdot 3^{n-1}}$
(c) $2n - \frac{1}{6} + \frac{1}{3\cdot 2^{n-1}}$ (d) $\frac{5}{3}n - \frac{7}{6} + \frac{1}{2\cdot 3^n - 1}$

24. The converse of the contrapositive of the conditional $p \to \sim q$ is
(a) $\sim q \to p$ (b) $\sim p \to \sim q$
(c) $\sim p \to q$ (d) $p \to q$

25. If set $A = \{1, 2, 3, 4, 5\}$, then the total number of unordered pairs of disjoint subsets of A is equal to
(a) 102 (b) 122
(c) 120 (d) 112

26. If $P = \begin{bmatrix} a & 7 \\ 3 & a \end{bmatrix}$ and $|P^3| = 64$, then the value of a is
(a) ± 1 (b) ± 5
(c) ± 2 (d) ± 3

27. If $3x - y + 2z = 2$, $5x + 3y + \lambda = 7$ and $2x + y + 3z = -4$, then the value of λ such that system of equations has no solution, is
(a) 8 (b) 7
(c) 4 (d) 2

28. If a variable plane $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$ at a unit distance from origin cuts the coordinate axes at P, Q, R and centroid (x, y, z) satisfies the equation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = t$, then the value of t is
(a) 3 (b) 9
(c) $\frac{1}{9}$ (d) $\frac{1}{3}$

29. If the points $(a, b), (c, d)$ and $(a - c, b - d)$ are collinear, then $\frac{a}{c}$ equals
(a) $\frac{b}{d}$ (b) $\frac{b}{c}$ (c) $\frac{d}{b}$ (d) b

30. The general solution of differential equation $x\,dy - y\,dx - \sqrt{x^2 + y^2}\,dx = 0$ is
(a) $y + \sqrt{x^2 + y^2} = kx^2$ (b) $x + \sqrt{x^2 + y^2} = ky^2$
(c) $y + \sqrt{x^2 + y^2} = kx$ (d) None of these

Answers

1. (c)	2. (a)	3. (b)	4. (b)	5. (d)
6. (d)	7. (d)	8. (a)	9. (c)	10. (d)
11. (a)	12. (b)	13. (b)	14. (c)	15. (a)
16. (d)	17. (b)	18. (b)	19. (a)	20. (a)
21. (a)	22. (c)	23. (b)	24. (c)	25. (b)
26. (b)	27. (a)	28. (b)	29. (a)	30. (a)

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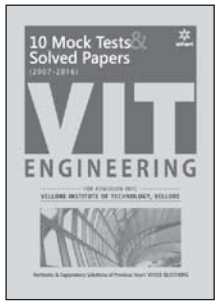

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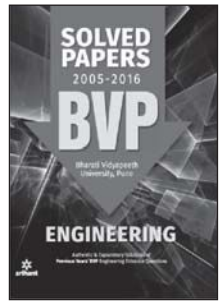

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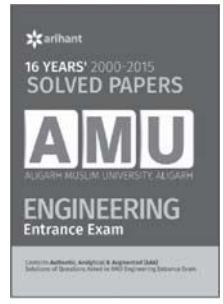

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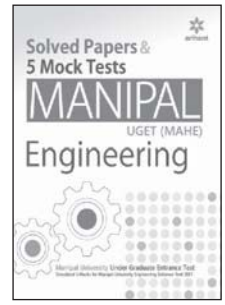

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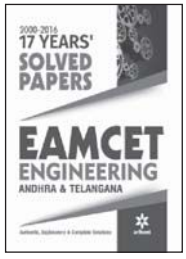

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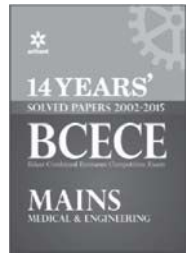

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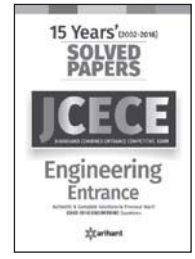

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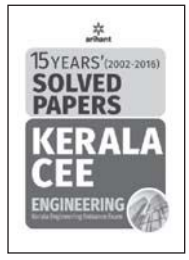

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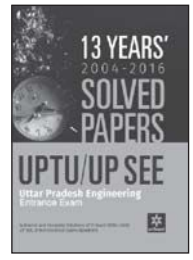

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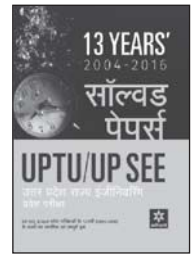

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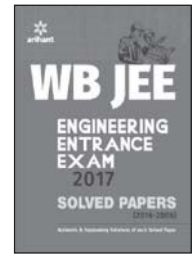

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PROBABILITY

A collection of chapterwise Statement type problems for JEE Main

Directions (Q. Nos. 1-30) Each question has four choices (a), (b), (c) and (d), out of which only one is correct. Each question contains Statement I and Statement II.

(a) If both statements are true and Statement II is the correct explanation of Statement I

(b) If both statements are true but Statement II is not the correct explanation of Statement I

(c) If Statement I is true and Statement II is false

(d) If Statement I is false and Statement II is true

1. **Statement I** 20 persons are sitting in a row. Two of these persons are selected at random. The probability that the two selected persons are not together, is 0.9.

Statement II If $\overline{A}$ denotes the negation of an event, then $P(\overline{A}) = 1 - P(A)$.

2. **Statement I** A natural number x is chosen at random from the first 100 natural numbers. The probability that $\frac{(x-10)(x-50)}{(x-30)} < 0$ is 0.28.

Statement II For any event A, $0 \le P(A) \le 1$.

3. **Statement I** The probability of drawing either a king or an ace from a pack of 52 playing cards is 2/13.

Statement II For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

4. Let A and B be two events such that $P(A \cup B) = P(A \cap B)$.

Statement I $P(A \cap \overline{B}) = P(\overline{A} \cap B) = 0$

Statement II $P(A) + P(B) = 1$

5. **Statement I** If A and B are two events such that $P(A) = 1$, then A and B are independent.

Statement II A and B are two independent events iff $P(A \cap B) = P(A) \cdot P(B)$.

6. Let A, B, C be three mutually independent events associated to a random experiment.

Statement I A and $B \cup C$ are independent.

Statement II A and $B \cap C$ are independent.

7. **Statement I** Let $n \ge 3$ and $A_1, A_2, A_3, \ldots, A_n$ be n independent events such that $P(A_k) = \frac{1}{k+1}$ for $1 \le k \le n$, then $P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \ldots \cap A_n) = \frac{1}{n+1}$.

Statement II Let $A_1, A_2, A_3, \ldots, A_n$ be $n\,(\ge 3)$ events associated to a random experiment. Then, $A_1, A_2, A_3, \ldots, A_n$ are independent iff $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2)\,P(A_3)\ldots P(A_n)$.

8. Let A, B and C be three events such that $P(C) = 0$.

Statement I $P(A \cap B \cap C) = 0$

Statement II $P(A \cup B \cup C) = P(A \cup B)$

9. There are two persons A and B such that chances of B speaking truth are twice that of A and A speaks truth in more than 25% cases.

Statement I If A and B contradict each other in narrating the same statement with probability 1/2, then it is certain that B never tells a lie.

Statement II The probability that A speaks truth is 1/2.

10. Let A and B be two independent events.

Statement I If $P(A) = 0.3$ and $P(A \cup \overline{B}) = 0.8$, then $P(B) = \frac{2}{7}$.

Statement II For any event E, $P(\overline{E}) = 1 - P(E)$.

11. **Statement I** The probability of solving a new problem by 3 students are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. The probability that the problem will be solved by them, is $\frac{3}{4}$.

Statement II If A, B and C are three independent events, then the probability that atleast one of them happens is $1 - P(\overline{A})\,P(\overline{B})\,P(\overline{C})$.

12. Four members are chosen at random (without replacement) from the set {1, 2, 3, …, 20}.

Statement I The probability that the chosen number when arranged in some order form an AP, is 1/85.

Statement II If the four chosen numbers form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$.

13. A fair die is thrown twice. Let (a, b) denotes the outcome in which the first throw shows a and the second throw shows b. Let A and B be the events such that $A = \{(a,b): a \text{ is even}\}$, $B = \{(a,b): b \text{ is even}\}$.

Statement I If $C = \{(a,b): a+b \text{ is odd}\}$, then $P(A \cap B \cap C) = \frac{1}{8}$.

Statement II If $D = \{(a,b): a+b \text{ is even}\}$, then $P\left(\frac{A \cap B \cap D}{A \cup B}\right) = \frac{1}{3}$.

14. Consider the system of equations $ax + by = 0$ and $cx + dy = 0$, where $a, b, c, d \in \{0,1\}$.

Statement I The probability that the system of equations has a unique solution, is $\frac{3}{8}$.

Statement II The probability that the system has a solution, is 1.

15. Let $H_1, H_2, H_3, \ldots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$; $i = 1, 2, 3, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement I $P(H_i/E) > P(E/H_i)\,P(H_i)$ for $i = 1, 2, 3, \ldots, n$.

Statement II $\sum_{i=1}^{n} P(H_i) = 1$

16. **Statement I** The probability of getting a tail most of the times in 10 tosses of an unbiased coin, is $\frac{1}{2}\left\{1 - \frac{10!}{2^{10}5!5!}\right\}$.

Statement II ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \ldots + {}^{2n}C_n = 2^{2n-1}, n \in N$

17. **Statement I** For events A and B of sample space, if $P\left(\frac{A}{B}\right) \geq P(A)$, then $P\left(\frac{B}{A}\right) \geq P(B)$.

Statement II $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$, where $P(B) \neq 0$

18. A fair die is rolled once.

Statement I The probability of getting a composite number is 1/3.

Statement II There are three possibilities for the obtained number:

(i) The number is a prime number.

(ii) The number is a composite number.

(iii) The number is 1.

Hence, probability of getting a prime number is 1/3.

19. **Statement I** If $P(A) = 0.25$, $P(B) = 0.50$ and $P(A \cap B) = 0.14$, then the probability that neither A nor B occurs is 0.39.

Statement II $\overline{A \cup B} = \overline{A} \cup \overline{B}$

20. Let A and B be two events such that $P(A) = 3/5$ and $P(B) = 2/3$.

Statement I $4/15 \leq P(A \cap B) \leq 3/5$

Statement II $2/5 \leq P(A/B) \leq 9/10$

21. **Statement I** Out of 5 tickets consecutively numbered, three are drawn at random. The chance that the numbers on them are in AP, is 2/5.

Statement II Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random. The chance that the numbers on them are in AP, is $3n/(4n^2 - 1)$.

22. Let A and B be two events such that $P(A \cup B) \geq 3/4$ and $1/8 \leq P(A \cap B) \leq 3/8$.

Statement I $P(A) + P(B) \geq 7/8$

Statement II $P(A) + P(B) \leq 11/8$

23. **Statement I** The probability of drawing either an ace or a king from a pack of card in a single draw is 2/13.

Statement II For two events A and B which are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

24. Let A and B be two independent events.

Statement I If $P(A) = 0.4$ and $P(A \cup \overline{B}) = 0.9$, then $P(B) = 1/6$.

Statement II If A and B are independent events, then $P(A \cap B) = P(A)\,P(B)$.

25. Consider an event for which probability of success is 1/2.

Statement I Probability that in n trials, there are r success, where $r = 4K$ and K is an integer, is $\frac{1}{4} + \frac{1}{2^{(n/2)+1}} \cos\left(\frac{n\pi}{4}\right)$.

Statement II ${}^nC_0 + {}^nC_4 + {}^nC_8 + \ldots = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$

26. **Statement I** If A and B are two events such that $0 < P(A)$ and $P(B) < 1$, then $P(A/\overline{B}) + P(\overline{A}/\overline{B}) = 3/2$.

Statement II If A and B are two events such that $0 < P(A)$ and $P(B) < 1$, then $P(A/B) = P(A \cap B)/P(B)$ and $P(\overline{B}) = P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})$.

27. **Statement I** If a fair coin is tossed 15 times, then the probability of getting head as many times in the first ten throws as in the last five is 3003/32768.

Statement II Sum of the series ${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + \ldots + {}^mC_0\,{}^nC_r = {}^{m+n}C_r$

28. **Statement I** If $A=\{2,4,6\}$ and $B=\{1,2,3\}$, where A and B are the events of numbers occurring on a die, then $P(A)+P(B)=1$.

Statement II If $A_1, A_2, A_3, \ldots, A_n$ are all mutually exclusive events, then $P(A_1)+P(A_2)+P(A_3)+\ldots+P(A_n)=1$.

29. **Statement I** There are 4 addressed envelopes and 4 letters for each one of them. The probability that no letter is mailed in its correct envelope is 3/8.

Statement II The probability that all letters are not mailed in their correct envelope is 23/24.

30. **Statement I** If A, B, C are three mutually independent events, then A and $B\cup C$ are also independent events.

Statement II Two events A and B are independent if and only if $P(A\cap B)=P(A)\,P(B)$.

Answers with Explanation

1. ***(a)*** Clearly, Statement II is true.
The number of ways of selecting 2 persons out of 20 persons sitting in a row is ${}^{20}C_2(=190)$ and the number of ways in which two selected persons sit together is 19.
Let A denotes the event "Two selected persons sit together". Then, $P(A)=\frac{19}{190}=\frac{1}{10}$

$\therefore$ Required probability $=P(\overline{A})$
$=1-P(A)$
$=1-\frac{1}{10}$
$=\frac{9}{10}=0.9$

2. ***(b)*** Clearly, Statement II is true.
Now, $\frac{(x-10)(x-50)}{(x-30)}<0$

$\Rightarrow \quad 0<x<10$ or $30<x<50$
$\Rightarrow x=1, 2, 3, 4, 5, 6, 7, 8, 9$ or $x=31, 32, \ldots, 49$.
So, x takes 28 values.
$\therefore$ Probability of given event $=\frac{28}{100}=0.28$

3. ***(a)*** Clearly, Statement II is true.
Considering the following events
$A=$ Getting a king in a draw,
$B=$ Getting an ace in a draw.
Clearly, $P(A)=\frac{1}{13}, P(B)=\frac{1}{13}, P(A\cap B)=0$

$\therefore$ Required probability $=P(A\cup B)$
$=P(A)+P(B)-P(A\cap B)$
$=\frac{1}{13}+\frac{1}{13}-0=\frac{2}{13}$

4. ***(c)*** We have, $P(A\cup B)=P(A\cap B)$

$\Rightarrow \quad P(A)+P(B)-P(A\cap B)=P(A\cap B) \quad \ldots(i)$
$\Rightarrow \quad \{P(A)-P(A\cap B)\}+\{P(B)-P(A\cap B)\}=0$
$\Rightarrow \quad P(A\cap\overline{B})+P(\overline{A}\cap B)=0$
$\therefore$ Statement I is true.
Also, $P(A)+P(B)=2P(A\cap B)$ [from Eq. (i)]
So, $P(A)+P(B)=1$ may not be true.

5. ***(a)*** Clearly, Statement II is true.
If $P(A)=1$, then $P(\overline{A})=0$
Now, $\overline{A}\cap B\subset\overline{A}$
$\Rightarrow \quad P(\overline{A}\cap B)\le P(\overline{A})$
$\Rightarrow \quad P(\overline{A}\cap B)\le 0$
$\Rightarrow \quad P(\overline{A}\cap B)=0$
$\Rightarrow \quad P(\overline{A}\cap B)=P(\overline{A})\,P(B) \quad [\because P(\overline{A})=0]$
$\therefore\ \overline{A}$ and B are independent events.
Hence, A and B are independent events.

6. ***(b)*** It is given that A, B, C are three mutually independent events.
$\therefore P(A\cap B)=P(A)\,P(B),\ P(B\cap C)=P(B)\,P(C),$
$P(C\cap A)=P(C)\,P(A)$
and $P(A\cap B\cap C)=P(A)\,P(B)\,P(C)$
Now, $P(A\cap(B\cup C))=P\{(A\cap B)\cup(A\cap C)\}$
$=P(A\cap B)+P(A\cap C)-P\{(A\cap B)\cap(A\cap C)\}$
$=P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)$
$=P(A)\,P(B)+P(A)\,P(C)-P(A)\,P(B)\,P(C)$
$=P(A)\{P(B)+P(C)-P(B)\,P(C)\}$
$=P(A)\,P(B\cup C)$
$\therefore\ A$ and $B\cup C$ are independent events.
Also, $(A\cap B\cap C)=P(A)\,P(B)\,P(C)$
$=P(A)\,P(B\cap C)$
$\therefore\ A$ and $B\cap C$ are independent events.

7. ***(c)*** We know that three or more events associated to a random experiment are independent iff they are pairwise independent as well as independent together.
So, Statement II is not true.
If $P(A_k)=\frac{1}{k+1}$ for $k=1, 2, 3, \ldots, n$ and $A_1, A_2, A_3, \ldots, A_n$ are independent, then

$P(\overline{A}_1\cap\overline{A}_2\cap\overline{A}_3\cap\ldots\cap\overline{A}_n)$
$=P(\overline{A}_1)\,P(\overline{A}_2)\,P(\overline{A}_3)\ldots P(\overline{A}_n)$

$\Rightarrow \quad P(\overline{A}_1\cap\overline{A}_2\cap\overline{A}_3\cap\ldots\cap\overline{A}_n)=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{n}{n+1}$
$=\frac{1}{n+1}$

So, Statement I is true.

8. ***(b)*** We know that,
$A\cap B\cap C\subseteq C$
$\therefore \quad P(A\cap B\cap C)\le P(C)=0$
$\Rightarrow \quad P(A\cap B\cap C)=0$
So, Statement I is true.
Similarly, $A\cap C\subseteq C$ and $B\cap C\subseteq C$
$\Rightarrow \quad P(A\cap C)=0$ and $P(B\cap C)=0$
$\therefore P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)$
$-P(C\cap A)+P(A\cap B\cap C)$
$=P(A)+P(B)-P(A\cap B)=P(A\cup B)$
So, Statement II is true.

9. ***(a)*** Consider the following events
$E = A$ speaks truth; $F = B$ speaks truth
Let $P(E) = p$. Then $P(F) = 2p$
It is given that

$$P\{(E \cap \overline{F})\} \cup (\overline{E} \cap F)\} = \frac{1}{2}$$

$$\Rightarrow \quad P(E \cap \overline{F}) + P(\overline{E} \cap F) = \frac{1}{2}$$

$$\Rightarrow \quad P(E)\,P(\overline{F}) + P(\overline{E})\,P(F) = \frac{1}{2}$$

$$\Rightarrow \quad p(1-2p) + (1-p)\,2p = \frac{1}{2}$$

$$\Rightarrow \quad p - 2p^2 + 2p - 2p^2 = \frac{1}{2}$$

$$\Rightarrow \quad 8p^2 - 6p + 1 = 0$$

$$\Rightarrow \quad (4p-1)(2p-1) = 0$$

$$\Rightarrow \quad p = \frac{1}{2} \qquad \left[\because p \neq \frac{1}{4}\right]$$

$$\therefore \quad P(E) = \frac{1}{2}$$

So, Statement II is true.
Now, $P(E) = \frac{1}{2} \Rightarrow P(F) = 2p = 1$
$\Rightarrow F$ is a certain event.
So, B never tells a lie.
Hence, Statement I is true, Statement II is true and Statement II is the correct explanation of Statement I.

10. ***(a)*** Clearly, Statement II is true.
If A and B are independent events, then

$$P(A \cup \overline{B}) = 0.8$$

$$\Rightarrow \quad 1 - P(\overline{A})P(B) = 0.8$$

$$\Rightarrow \quad 1 - (1 - P(A))P(B) = 0.8$$

$$\Rightarrow \quad 1 - (0.7)P(B) = 0.8$$

$$\Rightarrow \quad P(B) = \frac{2}{7}$$

Hence, Statement I is true, Statement II is true and Statement II is the correct explanation of Statement I.

11. ***(a)*** Clearly, Statement II is true.
Required probability $= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}$
Hence, Statement I is true, Statement II is true and Statement II is the correct explanation of Statement I.

12. ***(c)*** Total number of ways of selecting 4 numbers out of first 20 natural numbers $= {}^{20}C_4$
Number of AP's with common difference 1 is 17.
(i.e. 1, 2, 3, 4; 2, 3, 4, 5; …; 17, 18, 19, 20)
Number of AP's with common difference 2 is 14.
(i.e. 1, 3, 5, 7; 2, 4, 6, 8;…; 14, 16, 18, 20)
Number of AP's with common difference 3 is 11.
(i.e. 1, 4, 7, 10; 2, 5, 8, 11; ... ; 11, 14, 17, 20)
and so on.
The maximum common difference can be 6 and the number of such AP's is 2. (i.e. 1, 7, 13, 19; 2, 8, 14, 20)
$\therefore$ Total number of ways in which chosen numbers are in AP
$= 17 + 14 + 11 + \ldots + 2 = 57$
$\therefore$ Probability that the chosen numbers are in AP

$$= \frac{57}{{}^{20}C_4} = \frac{1}{85}$$

Clearly, the set of all possible values of common difference is $\{\pm1, \pm2, \pm3, \pm4, \pm5, \pm6\}$.
So, Statement II is not true.

13. ***(d)*** We have, $A \cap B = \{(a, b) : a$ and b both are even$\}$

$$\Rightarrow \quad A \cap B \cap C = \phi$$

$$\Rightarrow \quad P(A \cap B \cap C) = 0$$

So, Statement I is not true.
Clearly, $A \cap B \subseteq D$

$$\therefore \quad A \cap B \cap D = A \cap B$$

$$\Rightarrow \quad P\left(\frac{A \cap B \cap D}{A \cup B}\right) = P\left(\frac{A \cap B}{A \cup B}\right)$$

$$\Rightarrow \quad P\left(\frac{A \cap B \cap D}{A \cup B}\right) = \frac{P(A \cap B)}{P(A \cup B)}$$

$$\Rightarrow \quad P\left(\frac{A \cap B \cap D}{A \cup B}\right) = \frac{P(A \cap B)}{P(A) + P(B) - P(A \cap B)}$$

$$\Rightarrow \quad P\left(\frac{A \cap B \cap D}{A \cup B}\right) = \frac{1/4}{\frac{1}{2} + \frac{1}{2} - \frac{1}{4}} = \frac{1}{3}$$

So, Statement II is true.

14. ***(b)*** The given system of equations is a homogeneous system of equations which is always consistent. So, the probability that the system has a solution, is 1.
So, Statement II is true.
The given system of equations will have a unique solution iff $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

As $a, b, c, d \in \{0, 1\}$. So, each of a, b, c and d can assume two values. Therefore, there are 2^4 sets of values of a, b, c and d.
Clearly, $ad - bc \neq 0$ iff $ad = 1$ and $bc = 0$ or $ad = 0$ and $bc = 1$.
Now, $ad = 1$ and $bc = 0$ iff ($a = 1, d = 1$ and $b = 1, c = 0$) or ($a = 1, d = 1$ and $b = 0, c = 0$)
So, there are three sets of values of a, b, c and d satisfying $ad = 1$ and $bc = 0$.
Similarly, there are three sets of values of a, b, c and d satisfying $bc = 1$ and $ad = 0$.
Thus, out of 2^4 sets of values of a, b, c and d. Therefore, six sets for which the given system has a unique solution.
$\therefore$ Probability that the system has a unique solution

$$= \frac{6}{16} = \frac{3}{8}$$

Hence, Statement I is true, Statement II is true but Statement II is not the correct explanation of Statement I.

15. ***(d)*** Since, $H_1, H_2, H_3, \ldots, H_n$ are exhaustive events.
$\therefore \bigcup_{i=1}^{n} H_i = S$, where S is the sample space.

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{n} H_i\right) = P(S) \quad \Rightarrow \quad \sum_{i=1}^{n} P(H_i) = 1$$

$[\because H_1, H_2, H_3, \ldots, H_n$ are mutually exclusive events$]$
So, Statement II is true.
Now, two cases arise :
***Case* I** When $H_i \cap E = \phi$ for some H_i

In this case, we have

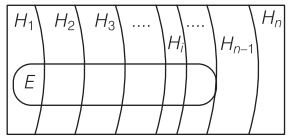
$$P(H_i \cap E) = 0 \text{ for some } i$$

$$\Rightarrow \quad P\left(\frac{E}{H_i}\right)P(H_i) = 0$$

and $P(H_i / E) = \dfrac{P(H_i \cap E)}{P(E)} = 0$ for some i

$\therefore P(H_i / E) = P(E / H_i)P(H_i)$ for some i

So, Statement I is not true for such values of i.



***Case* II** When $0 < P(H_i \cap E) < 1$ for $i = 1, 2, \ldots, n$

In this case, we have

$$P\left(\frac{H_i}{E}\right) = \frac{P(H_i \cap E)}{P(E)} = \frac{P(H_i)P(E / H_i)}{P(E)}$$

$$\Rightarrow \quad P\left(\frac{H_i}{E}\right) > P(H_i)P\left(\frac{E}{H_i}\right) \qquad [\because\ 0 < P(E) < 1]$$

Hence, Statement I is not always true.

16. ***(c)*** Let p be the probability of getting a tail in a single trial. Then, $p = \dfrac{1}{2} = q$.

Let X denotes the number of tails in 10 trials.

Then, $\quad P(X = r) = {}^{10}C_r\left(\frac{1}{2}\right)^{10}$

$\therefore$ Probability of getting a tail most of the times

$$= P(X \geq 6) = \sum_{r=6}^{10} P(X = r)$$

$$= \left(\frac{1}{2}\right)^{10}[{}^{10}C_6 + {}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}]$$

$$= \left(\frac{1}{2}\right)^{11}[{}^{10}C_0 + {}^{10}C_1 + \ldots + {}^{10}C_{10} - {}^{10}C_5]$$

$$= \left(\frac{1}{2}\right)^{11}\left(2^{10} - \frac{10!}{5!5!}\right) = \frac{1}{2}\left\{1 - \frac{10!}{2^{10}5!5!}\right\}$$

So, Statement I is true.

We have, ${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \ldots + {}^{2n}C_n + {}^{2n}C_{n+1} + \ldots + {}^{2n}C_{2n} = 2^{2n}$

$$\Rightarrow 2({}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \ldots + {}^{2n}C_{n-1}) + {}^{2n}C_n = 2^{2n}$$

$$\Rightarrow {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \ldots + {}^{2n}C_{n-1} = 2^{2n-1} - \frac{1}{2}{}^{2n}C_n$$

So, Statement II is not true.

17. ***(a)*** $P(A / B) \geq P(A) \Rightarrow \dfrac{P(A \cap B)}{P(B)} \geq P(A)$

$$\Rightarrow \frac{P(A \cap B)}{P(A)} \geq P(B) \Rightarrow P\left(\frac{B}{A}\right) \geq P(B)$$

18. ***(c)*** Statement I is true as there are six equally likely possibilities of which only two are favourable (4 and 6). Hence, probability that the obtained number is composite, is 2/6 i.e. 1/3. Statement II is not true, as the three possibilities are not equally likely.

19. ***(c)*** Required probability is

$$P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)] = 0.39$$

Clearly, Statement II is not true.

20. ***(b)*** $\because P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$\geq P(A) + P(B) - 1$$

$$\geq \frac{3}{5} + \frac{2}{3} - 1 \geq \frac{4}{15}$$

Also, $P(A \cap B) \leq P(A) \leq \dfrac{3}{5}$

$$\therefore \quad \frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$$

Also, $\quad \dfrac{4}{15P(B)} \leq \dfrac{P(A \cap B)}{P(B)} \leq \dfrac{3}{5P(B)}$

$$\Rightarrow \quad \frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$$

21. ***(a)*** $2n + 1 = 5 \Rightarrow n = 2$

$$\therefore \quad P(E) = \frac{3n}{4n^2 - 1} = \frac{6}{15} = \frac{2}{5}$$

As a, b, c are in AP, so $a + c = 2b$

Hence, $a + b$ is even.

Therefore, a and c both are even or both odd. So, the number of ways of choosing a and c is ${}^{n}C_2 + {}^{n+1}C_2 = n^2$.

$$\therefore \quad P(E) = \frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2 - 1}$$

22. ***(b)*** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\therefore \quad 1 \geq P(A) + P(B) - P(A \cap B) \geq \frac{3}{4}$$

$$\Rightarrow \quad P(A) + P(B) - \frac{1}{8} \geq \frac{3}{4}$$

[since, minimum value of $P(A \cap B)$ is 1/8]

$$\Rightarrow \quad P(A) + P(B) \geq \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

$$\Rightarrow \quad 1 \geq P(A) + P(B) - \frac{3}{8}$$

[since, maximum value of $P(A \cap B)$ is 3/8]

$$\Rightarrow \quad P(A) + P(B) \leq 1 + \frac{3}{8} = \frac{11}{8}$$

23. ***(a)*** $P(\text{ace} \cup \text{king}) = P(\text{ace}) + P(\text{king}) - P(\text{ace} \cap \text{king})$

$$= \frac{1}{13} + \frac{1}{13} - 0 = \frac{2}{13}$$

24. ***(a)*** $P(A \cup \overline{B}) = 1 - P(\overline{A \cup \overline{B}})$

$$= 1 - P(\overline{A} \cap B) = 1 - P(\overline{A})\,P(B)$$

$$\Rightarrow \quad 0.9 = 1 - 0.6 \times P(B)$$

$$\Rightarrow \quad P(B) = \frac{1}{6}$$

25. ***(c)*** According to Statement I, required probability

$$= {}^{n}C_0\left(\frac{1}{2}\right)^n + {}^{n}C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{n-4} + {}^{n}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8} + \ldots$$

$$= \left(\frac{1}{2}\right)^n({}^{n}C_0 + {}^{n}C_4 + {}^{n}C_8 + \ldots)$$

Now, consider the binomial expansion,

$$(1 + x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots$$

On putting $x = i$, we get

$$(1+i)^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \ldots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \ldots)$$

$$\Rightarrow \left[\sqrt{2}\cos\left(\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \ldots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \ldots)$$

$$\Rightarrow {}^nC_0 - {}^nC_2 + {}^nC_4 - \ldots = 2^{n/2}\cos\frac{n\pi}{4}$$

Also, we know that,

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = 2^{n-1}$$

$$\Rightarrow \quad 2({}^nC_0 + {}^nC_4 + {}^nC_8 + \ldots) = 2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}$$

Hence, the required probability is

$$\frac{1}{4} + \frac{1}{2^{(n/2)+1}}\cos\left(\frac{n\pi}{4}\right).$$

26. ***(d)*** $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow \quad P(\overline{B}) = P((A \cup \overline{A}) \cap \overline{B}) = P((A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}))$$

Hence, Statement II is true.

Now, $$P\left(\frac{A}{\overline{B}}\right) + P\left(\frac{\overline{A}}{\overline{B}}\right) = \frac{P(A \cap \overline{B})}{P(\overline{B})} + \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$$

$$= \frac{P(A \cap \overline{B}) + P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{B})}{P(\overline{B})} = 1$$

Therefore, Statement I is not true.

27. ***(a)*** In the binomial theorem, we have proved Statement II. Now, there may be 0, 1, 2, 3, 4 or 5 heads in the last five throws and the same can be for the first ten throws.

Thus, number of cases may be given by

$$m = {}^5C_0\,{}^{10}C_0 + {}^5C_1\,{}^{10}C_1 + {}^5C_2\,{}^{10}C_2 + {}^5C_3\,{}^{10}C_3 + {}^5C_4\,{}^{10}C_4 + {}^5C_5\,{}^{10}C_5$$

$$= {}^5C_0\,{}^{10}C_{10} + {}^5C_1\,{}^{10}C_9 + {}^5C_2\,{}^{10}C_8 + {}^5C_3\,{}^{10}C_7 + {}^5C_4\,{}^{10}C_6 + {}^5C_5\,{}^{10}C_5$$

$$= {}^{10+5}C_{10} = {}^{15}C_{10} = 3003$$

Total number of ways $(n) = 2^{15} = 32768$.

Hence, the required probability is $m/n = \frac{3003}{32768}$.

28. ***(c)*** $P(A) + P(B) = 1$ is true as A and B are mutually exclusive and exhaustive events, but Statement II is false as it is not given that the events are exhaustive.

29. ***(b)*** The total number of cases, $n(S) = 4!$.

Let E be the event that no letter is mailed in its correct envelope. Then, the favourable number of cases

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

Hence, the required probability is $P(E) = \frac{9}{24} = \frac{3}{8}$

Also, the probability that all the letters are placed in the correct envelope is 1/24.

Thus, the probability that all the letters are not placed in the correct envelope is 23/24.

Hence, Statement II is correct but does not explain Statement I.

30. ***(a)*** $P\{A \cap (B \cap C)\} = P(A \cap B \cap C)$

$$= P(A)\,P(B)\,P(C)$$

$$\therefore P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

$$= P(A)P(B) + P(A)P(C) - P(A)\,P(B)\,P(C)$$

$$= P(A)\,[P(B) + P(C) - P(B)\,P(C)]$$

$$= P(A)\,P(B \cup C)$$

Therefore, A and $B \cup C$ are independent events.

Continued from Page 32

23. The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, is

(a) $x^2 + y^2 = a^2 + \lambda$ *(b)* $x^2 + y^2 = b^2 + \lambda$

(c) $x^2 + y^2 = a^2 + b^2 + \lambda$ *(d)* $x^2 + y^2 = a^2 + b^2$

24. If $S(3, 4)$ and $S'(9, 12)$ are two foci of an ellipse and the foot of the perpendicular from S on a tangent to the ellipse has the coordinates $(1, -4)$, then the eccentricity of the ellipse is

(a) $\frac{3}{13}$ *(b)* $\frac{4}{13}$

(c) $\frac{5}{13}$ *(d)* None of these

25. A line intersects the ellipse $\frac{x^2}{4a^2} + \frac{y^2}{a^2} = 1$ at A and B and the parabola $y^2 = 4a(x + 2a)$ at C and D. The line segment AB subtends a right angle at the centre of the ellipse. Then, the locus of the point of intersection of tangents to the parabola at C and D, is

(a) $y^2 - a^2 = \frac{5}{4}(x - 4a)^2$ *(b)* $y^2 - 2a^2 = 10(x - 4a)^2$

(c) $y^2 + a^2 = \frac{5}{2}(x - 4a)^2$ *(d)* $y^2 + 4a^2 = 5(x + 4a)^2$

Answers

1. (b)	**2.** (b)	**3.** (b)	**4.** (d)	**5.** (c)
6. (a)	**7.** (d)	**8.** (a)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (c)
16. (d)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (c)
21. (b)	**22.** (d)	**23.** (c)	**24.** (c)	**25.** (d)

CONCEPT MAP your Revision Tool

TRIGONOMETRY

1 DOMAIN AND RANGE OF TRIGONOMETRICAL FUNCTIONS

Function	Domain	Range
$\sin\theta$	R	$[-1, 1]$
$\cos\theta$	R	$[-1, 1]$
$\tan\theta$	$R-\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$	R
$\cot\theta$	$R-\{n\pi : n \in Z\}$	R
$\sec\theta$	$R-\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$	$R-(-1, 1)$
$\operatorname{cosec}\theta$	$R-\{n\pi : n \in Z\}$	$R-(-1, 1)$

2 SUM AND DIFFERENCE FORMULAE

Following are the trigonometric formulae for sum and difference of angles:

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(iii) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(iv) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(v) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(vi) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(vii) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

(viii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(ix) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$
$=\cos^2 B-\cos^2 A$

(x) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$
$=\cos^2 B-\sin^2 A$

3 T-RATIOS OF MULTIPLE ANGLES

(i) $\sin 2\theta=2\sin\theta\cos\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$
$=1-2\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1+\cos 2\theta=2\cos^2\theta, 1-\cos 2\theta=2\sin^2\theta$
or $\dfrac{1+\cos 2\theta}{2}=\cos^2\theta, \dfrac{1-\cos 2\theta}{2}=\sin^2\theta$

(iv) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(v) $\sin 3\theta=3\sin\theta-4\sin^3\theta$

(vi) $\cos 3\theta=4\cos^3\theta-3\cos\theta$

(vii) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

(viii) $\cos A\cos 2A\cos 2^2A\ldots\cos 2^{n-1}A=\dfrac{\sin 2^nA}{2^n\sin A}$

(ix) $\sin\theta\sin(60°-\theta)\sin(60°+\theta)=\dfrac{1}{4}\sin 3\theta$

(x) $\cos\theta\cos(60°-\theta)\cos(60°+\theta)=\dfrac{1}{4}\cos 3\theta$

(xi) $\tan\theta\tan(60°-\theta)\tan(60°+\theta)=\tan 3\theta$

4 SUM AND DIFFERENCE INTO PRODUCT

Following are the formulae to transform sum and difference into product:

(i) $\sin A+\sin B=2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(ii) $\sin A-\sin B=2\sin\left(\dfrac{A-B}{2}\right)\cos\left(\dfrac{A+B}{2}\right)$

(iii) $\cos A+\cos B=2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(iv) $\cos A-\cos B=-2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

(v) $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\cos B}$

(vi) $\tan A-\tan B=\dfrac{\sin(A-B)}{\cos A\cos B}$

(vii) $\cot A+\cot B=\dfrac{\sin(A+B)}{\sin A\sin B}$

(viii) $\cot A-\cot B=\dfrac{\sin(B-A)}{\sin A\sin B}$

5 PRODUCT INTO SUM OR DIFFERENCE

Following are the formulae to transform product into sum or difference:

(i) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(ii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(iii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(iv) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

6 T-RATIOS OF THE SUM OF THREE OR MORE ANGLES

(i) $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$
or $\sin(A+B+C)=\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A\tan B\tan C)$

(ii) $\cos(A+B+C)=\cos A\cos B\cos C-\sin A\sin B\cos C-\sin A\cos B\sin C-\cos A\sin B\sin C$
or $\cos(A+B+C)=\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)$

(iii) $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

(iv) $\sin(A_1+A_2+\ldots+A_n)=\cos A_1\cos A_2\ldots\cos A_n(S_1-S_3+S_5-S_7+\ldots)$

(v) $\cos(A_1+A_2+\ldots+A_n)=\cos A_1\cos A_2\ldots\cos A_n(1-S_2+S_4-S_6+\ldots)$

(vi) $\tan(A_1+A_2+\ldots+A_n)=\dfrac{S_1-S_3+S_5-S_7+\ldots}{1-S_2+S_4-S_6+\ldots}$

where, $S_1=\tan A_1+\tan A_2+\ldots+\tan A_n$
= Sum of the tangents of the separate angles,
$S_2=\tan A_1\tan A_2+\tan A_2\tan A_3+\ldots$
= Sum of the tangents taken two at a time
$S_3=\tan A_1\tan A_2\tan A_3+\tan A_2\tan A_3\tan A_4+\ldots$
= Sum of the tangents taken three at a time and so on.

(vii) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}=\dfrac{\sin\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$

(viii) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}=\dfrac{\cos\left\{\alpha+(n-1)\frac{\beta}{2}\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$

HEIGHTS AND DISTANCE

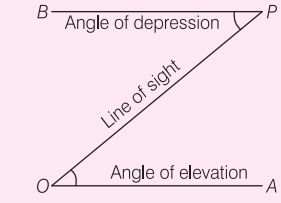
ANGLES OF ELEVATION AND DEPRESSION

Let O and P be two points such that the point P is at higher level. Let OA and PB be horizontal lines through O and P, respectively.

If an observer is at O and the point P is the object, then the line OP is called line of sight and $\angle AOP$ is known as the angle of elevation.

If an observer is at P and the point O is the object, then $\angle BPO$ is known as the angle of depression.

Obviously, the angle of elevation of a point P as seen from a point O is equal to the angle of depression of O as seen from P.



SOME USEFUL RESULTS

(i) Any line perpendicular to a plane is perpendicular to every line lying in the plane.

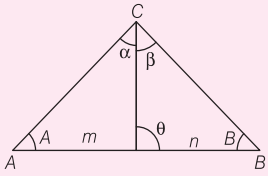
(ii) In a triangle, the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.

(iii) In an isosceles triangle, the median is perpendicular to the base.

(iv) Angles in the same segment of a circle are equal.

(v) ***m-n* Theorem**

In the given figure, we have



(a) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(b) $(m+n)\cot\theta = n\cot A - m\cot B$

7 MAXIMUM AND MINIMUM VALUES

The maximum and minimum values of trigonometrical expressions of the form $a\sin x + b\cos x$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$, respectively.

8 DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Function	Domain	Range
$\sin^{-1}x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

9 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Following are the properties of inverse trigonometric functions:

(A) PROPERTY I

(i) $\sin^{-1}(\sin\theta)=\theta, \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta)=\theta, \theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot\theta)=\theta, \theta\in(0,\pi)$

(v) $\sec^{-1}(\sec\theta)=\theta, \theta\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(vi) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta, \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

(B) PROPERTY II

(i) $\sin(\sin^{-1}x)=x, x\in[-1,1]$

(ii) $\cos(\cos^{-1}x)=x, x\in[-1,1]$

(iii) $\tan(\tan^{-1}x)=x, x\in R$

(iv) $\cot(\cot^{-1}x)=x, x\in R$

(v) $\sec(\sec^{-1}x)=x, x\in R-(-1,1)$

(vi) $\operatorname{cosec}(\operatorname{cosec}^{-1}x)=x, x\in R-(-1,1)$

(C) PROPERTY III

(i) $\sin^{-1}(-x)=-\sin^{-1}x, x\in[-1,1]$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x, x\in[-1,1]$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x, x\in R$

(iv) $\cot^{-1}(-x)=\pi-\cot^{-1}x, x\in R$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}x, x\in R-(-1,1)$

(vi) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x, x\in R-(-1,1)$

(D) PROPERTY IV

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}x, x\in R-(-1,1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec^{-1}x, x\in R-(-1,1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x, & x\in(0,\infty)\\ -\pi+\cot^{-1}x, & x\in(-\infty,0)\end{cases}$

(E) PROPERTY V

(i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}, x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}, x\in R$

(iii) $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}, x\in R-(-1,1)$

(F) PROPERTY VI

(i) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}$

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1), 0\le x\le 1$

(iii) $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), -1<x<1$

(G) PROPERTY VII

(i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3), -\frac{1}{2}\le x\le\frac{1}{2}$

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x), \frac{1}{2}\le x\le 1$

(iii) $3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

(H) PROPERTY VIII

(i) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right), -1\le x\le 1$

(ii) $2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0\le x<\infty$

(I) PROPERTY IX

(i) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1$

(ii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right), xy>-1$

(iii) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}$, $-1\le x, y\le 1, x^2+y^2\le 1$

(iv) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\}$, $-1\le x, y\le 1, x^2+y^2\le 1$

(v) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$, $-1\le x, y\le 1, x+y\ge 0$

(vi) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}$, $-1\le x, y\le 1, x\le y$

(J) PROPERTY X

(i) $\sin^{-1}x=\cos^{-1}(\sqrt{1-x^2})$

$=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$=\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1}x=\sin^{-1}(\sqrt{1-x^2})$

$=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$=\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$=\sec^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)$

$=\sec^{-1}(\sqrt{1+x^2})$

$=\operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

@ CLASS XII SYLLABUS

Rapid CONCEPT REVISION

TARGET 2017

INDEFINITE INTEGRATION

Integration is defined as an inverse process of differentiation. If the differential coefficient of a function $F(x)$ is $f(x)$, i.e. if $\frac{d}{dx}(F(x)) = f(x)$, then $\int f(x)\,dx = F(x) + C$, where C is called the constant of integration and $F(x)$ is called anti-derivative or primitive or integral of function $f(x)$.

Elementary Integration

Following are fundamental integration formulae:

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$

(ii) $\int \frac{1}{x}\,dx = \log_e |x| + C$

(iii) $\int e^x dx = e^x + C$

(iv) $\int a^x dx = \frac{a^x}{\log_e a} + C = a^x \log_a e + C$

(v) $\int \sin x\,dx = -\cos x + C$

(vi) $\int \cos x\,dx = \sin x + C$

(vii) $\int \tan x\,dx = \log|\sec x| + C = -\log|\cos x| + C$

(viii) $\int \cot x\,dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(ix) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

$= -\log|\sec x - \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(x) $\int \operatorname{cosec} x\,dx = -\log|\operatorname{cosec} x + \cot x| + C$

$= \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\left(\frac{x}{2}\right)\right| + C$

(xi) $\int \sec x \tan x\,dx = \sec x + C$

(xii) $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$

(xiii) $\int \sec^2 x\,dx = \tan x + C$

(xiv) $\int \operatorname{cosec}^2 x dx = -\cot x + C$

(xv) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C = -\cos^{-1} x + C$

(xvi) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C = -\cot^{-1} x + C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C = -\operatorname{cosec}^{-1} x + C$

Properties of Indefinite Integration

If $f(x)$ and $g(x)$ are two functions of a variable x, then

(i) $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$

(ii) $\int \left(\frac{d}{dx} f(x)\right) dx = f(x) + C$

(iii) $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.

(iv) $\int \{f(x) \pm g(x)\}\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Some Important Results

(i) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(iii) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iv) $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log|x + \sqrt{x^2 + a^2}| + C$

(vi) $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log|x + \sqrt{x^2 - a^2}| + C$

(vii) $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$

(viii) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(ix) $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log|x + \sqrt{a^2 + x^2}| + C$

(x) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + C$

Note If $\int f(x)\,dx = F(x) + C$, then $\int f(ax + b)\,dx = \frac{F(ax + b)}{a} + C$

Integration by Substitutions

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the type

$$I = \int f(g(x)) \cdot g'(x)\,dx$$

We substitute $g(x) = t$ and $g'(x)\,dx = dt$. This substitution reduces the integral to $\int f(t)\,dt$, which can be integrate easily. After evaluating this integral, we substitute the value of t in the result.

POINTS TO REMEMBER

- $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C\ (n \neq -1)$
- $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$
- $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$

Integration using Trigonometric Identities

When the integrand involves some trigonometric functions, then some known trigonometric identities are used to solve integral easily.

For this, following important trigonometric identities will be used:

(i) $\sin 2x = 2\sin x\cos x = \frac{2\tan x}{1 + \tan^2 x}$

(ii) $\cos 2x = \cos^2 x - \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$

$= 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(iii) $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

(iv) $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$

(v) $\sin 3x = -\sin^3 x + 3\cos^2 x\sin x = -4\sin^3 x + 3\sin x$

(vi) $\cos 3x = \cos^3 x - 3\sin^2 x\cos x = 4\cos^3 x - 3\cos x$

(vii) $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

(viii) $\cot 3x = \frac{3\cot x - \cot^3 x}{1 - 3\cot^2 x}$

(ix) $\tan\frac{x}{2} = \operatorname{cosec} x - \cot x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$

$= \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$

(x) $\cot\frac{x}{2} = \operatorname{cosec} x + \cot x = \sqrt{\frac{1 + \cos x}{1 - \cos x}}$

$= \frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$

(xi) $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

(xii) $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$

(xiii) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

(xiv) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

(xv) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(xvi) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

(xvii) $\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$

(xviii) $\sin A - \sin B = 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$

(xix) $\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$

(xx) $\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$

(xxi) Inverse trigonometric functions

(a) $\sin^{-1}(\sin x) = x$

(b) $\cos^{-1}(\cos x) = x$

(c) $\tan^{-1}(\tan x) = x$

(d) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$

(e) $\sec^{-1}(\sec x) = x$

(f) $\cot^{-1}(\cot x) = x$

SPEED BOOSTER

- $\sin^2 nx = \frac{1-\cos 2nx}{2}$
- $\cos^2 nx = \frac{1+\cos 2nx}{2}$
- $\sin nx = 2\sin\frac{nx}{2}\cos\frac{nx}{2}$
- $\sin^3 nx = \frac{3}{4}\sin nx - \frac{1}{4}\sin 3nx$
- $\cos^3 nx = \frac{3}{4}\cos nx + \frac{1}{4}\cos 3nx$

Some Standard Substitutions

The following substitutions are very useful to evaluate the integral involving the following expressions:

S.No.	Expression	Substitution
1.	$(a^2 + x^2), \sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}$	$x = a\tan\theta$ or $a\cot\theta$
2.	$(a^2 - x^2), \sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}$	$x = a\sin\theta$ or $a\cos\theta$
3.	$(x^2 - a^2), \sqrt{x^2 - a^2}, \frac{1}{\sqrt{x^2 - a^2}}$	$x = a\sec\theta$ or $a\ \mathrm{cosec}\,\theta$
4.	$\frac{2x}{a^2 - x^2}, \frac{2x}{a^2 + x^2}, \frac{a^2 - x^2}{a^2 + x^2}$	$x = a\tan\theta$
5.	$\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
6.	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$
7.	$\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}, \frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$ or $a\cot^2\theta$
8.	$\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}, \frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$ or $a\cos^2\theta$
9.	$\sqrt{\frac{x}{x-a}}, \sqrt{\frac{x-a}{x}}, \sqrt{x(x-a)}, \frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$ or $a\ \mathrm{cosec}^2\theta$
10.	$\sqrt{\frac{x-\alpha}{\beta-x}}, \sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
11.	$(x \pm \sqrt{x^2 \pm a^2})^n$	Expression inside the bracket $= t$
12.	$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}\ (n \in N, n > 1)$	$\frac{x+a}{x+b} = t$

Integration by Parts

If u and v are the differentiable functions of x, then

$$\int (u \cdot v)dx = u\int v\,dx - \int\left[\frac{du}{dx}\left(\int v\,dx\right)\right]dx$$

= (first function) × (integral of second function) – integral of [(derivative of first function) × (integral of second function)]

POINTS TO REMEMBER

- If in the product of two functions, one of the functions is not directly integrable (e.g. $\log x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, ...,), then we take it as the first function and the remaining function is taken as the second function.
- Usually, we consider that function as Ist function, which comes first in the word ILATE, where I stands for inverse circular function, L stands for logarithmic function, A stands for algebraic function, T stands for trigonometric function and E stands for exponential function.
- For the integration of logarithmic or inverse trigonometric functions alone, take unity (1) as the second function.
- If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

Some Important Results

Following are some important results based upon integration by parts.

(i) $\int e^x\{f(x) + f'(x)\}\,dx = e^x\ f(x) + C$

(ii) $\int e^{ax}\sin(bx + c)\,dx = \frac{e^{ax}}{a^2 + b^2}\{a\sin(bx + c) - b\cos(bx+c)\} + K$

(iii) $\int e^{ax}\cos(bx + c)\,dx = \frac{e^{ax}}{a^2 + b^2}\{a\cos(bx + c) + b\sin(bx + c)\} + K$

(iv) $\int\{xf'(x) + f(x)\}\,dx = x\,f(x) + C$

(v) $\int e^{ax}\left\{f(x) + \frac{f'(x)}{a}\right\}dx = e^{ax}\ \frac{f(x)}{a} + C$

(vi) $\int e^x\ \{f(x) - f''(x)\}\,dx = e^x\ \{f(x) - f'(x)\} + C$

(vii) $\int e^x\{f(x) + f'''(x)\}\,dx = e^x\ \{f(x) - f'(x) + f''(x)\} + C$

Integration of Rational Function using Partial Fractions

If a function is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)\ (\neq 0)$ are polynomials of degree m and n respectively.

Then,

(i) if $m < n$, this fraction is proper fraction.

(ii) if $m > n$, this fraction is improper fraction.

If fraction is improper, then it must be converted into proper fraction. To convert an improper fraction into a proper fraction, we divide $P(x)$ by $Q(x)$ such that $\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)}$, where $D(x)$ and $R(x)$ are polynomials and degree of $R(x)$ is less than degree of $Q(x)$. Any proper fractions can be expressed as the sum of rational function, if each having a simple factor of denominator. Each such fraction is called partial fraction and process to obtained them is called decomposition of proper rational fractions and it depends on nature of the factors of a denominator.

Rules for Partial Fractions

Here, some proper fractions and their corresponding partial fractions are given below:

S.No.	Types of proper rational functions	Partial fractions
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
3.	$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$
6.	$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where (x^2+ax+b) and (x^2+cx+d) cannot be factorised	$\frac{Ax+B}{x^2+ax+b}+\frac{Cx+D}{x^2+cx+d}$

Integration of Rational, Irrational, Algebraic and Transcendental Function

The following table will help you, to evaluate integral of some particular rational, irrational, algebraic and transcendental function.

S.No.	Types of the integral	Approach to solve
1.	$\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	Express ax^2+bx+c as sum or difference of two squares, i.e. $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$
2.	$\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$	Put $px+q = \lambda \cdot \frac{d}{dx}(ax^2+bx+c)+\mu$. Now, find the values of λ and μ and then integrate.
3.	$\int \frac{p(x)}{ax^2+bx+c}dx$	When $p(x)$ is a polynomial of degree 2 or more carry out the division and express in the form $\frac{p(x)}{ax^2+bx+c} = Q(x)+\frac{R(x)}{ax^2+bx+c}$, where $R(x)$ is a linear expression or constant, then integral reduces to the form discussed above.
4.	$\int \frac{x^2+a^2}{x^4+kx^2+a^4}dx$ or $\int \frac{x^2-a^2}{x^4+kx^2+a^4}dx$	After dividing both numerator and denominator by x^2. Put $x-\frac{a^2}{x}=t$ or $x+\frac{a^2}{x}=t$
5.	$\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x}, \int \frac{dx}{a\sin^2 x+b\cos^2 x+c}, \int \frac{dx}{(a\sin x+b\cos x)^2}$	To evaluate these types of integral, use following steps: ***Step* I** Divide numerator and denominator by $\cos^2 x$. ***Step* II** Replace $\sec^2 x$ (if any) in denominator by $1+\tan^2 x$ ***Step* III** Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$, simplify.
6.	$\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}, \int \frac{dx}{a\sin x+b\cos x}, \int \frac{dx}{a\sin x+b\cos x+c}$	To evaluate these types of integral, proceed as follows: ***Step* I** Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ ***Step* II** Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$. ***Step* III** Put $\tan\frac{x}{2} = t$ and then simplify.
7.	$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$	Put $a\sin x+b\cos x = \lambda \cdot \frac{d}{dx}(c\sin x+d\cos x)+\mu(c\sin x+d\cos x)$, then integral reduces to $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx = \lambda\int \frac{c\cos x-d\sin x}{c\sin x+d\cos x}dx+\mu\int \frac{c\sin x+d\cos x}{c\sin x+d\cos x}dx$

S.No.	Types of the integral	Approach to solve
8.	$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$	Put $a\sin x + b\cos x + c = \lambda\left\{\frac{d}{dx}(p\sin x + q\cos x + r)\right\} + \mu(p\sin x + q\cos x + r) + r$, then integral reduces to $\lambda\int \frac{p\cos x - q\sin x}{p\sin x + q\cos x + r}dx + \mu\int \frac{p\sin x + q\cos x + r}{p\sin x + q\cos x + r}\,dx + r\int \frac{dx}{p\sin x + q\cos x + r}$
9.	$\int\sqrt{\tan x}\,dx, \int\sqrt{\cot x}\,dx, \int \frac{dx}{\sin^4 x + \cos^4 x}, \int \frac{dx}{\sin^6 x + \cos^6 x}, \int \frac{\pm\sin x \pm \cos x}{a + b\sin x\cos x}\,dx$	Put $\tan x = t^2 \Rightarrow sec^2 x = 2t\,dt$, then integral reduces to the form 4.
10.	$\int \frac{dx}{(Ax + B)\sqrt{Cx + D}}$	Substitute $Cx + D = t^2$, then substitution will reduce the given integral into $\int \frac{2dt}{At^2 - AD + BC}$
11.	$\int \sin^n x \cos^n x\,dx$	To evaluate this type of integral, we use following substitution. (i) If m is an odd positive integer, put $\cos x = t$. (ii) If n is an odd positive integer, put $\sin x = t$. (iii) If m and n are both positive odd integers, put $\sin x = t$ or $\cos x = t$. (iv) If neither m nor n is an odd positive integer, then we look for their sum $m + n$. (a) If $m + n$ = negative even integer, then convert the given integrand in terms of $\tan x$ and $\sec^2 x$ and then put $\tan x = t$. (b) f m and n are small even integers, then convert them in terms of multiple angles by using the appropriate formula. (c) If m and n are large even positive integers, then change them in multiple angles with the help of Euler's formula in complex numbers.
12.	$\int \frac{dx}{(Ax^2 + B)\sqrt{Cx + D}}$	Substitute $Cx + D = t^2$, then substitution will reduce the given integral into $\int \frac{2C\,dt}{At^4 - 2DAt^2 + (AD^2 + BC^2)}$
13.	$\int \frac{dx}{(x - k)^r\sqrt{ax^2 + bx + c}}$	Substitute $x - k = \frac{1}{t}$, then the substitution will reduce the given integral into $\int \frac{t^{r-1}}{At^2 + Bt + C}\,dt$
14.	$\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}}$	Substitute $x = \frac{1}{t}$, then $\int \frac{dx}{(Ax^2 + B)\sqrt{Cx^2 + D}} = \int \frac{-\frac{1}{t^2}\,dt}{\left(\frac{A}{t^2} + B\right)\sqrt{\frac{C}{t^2} + D}}$ $= -\int \frac{t\,dt}{(A + Bt^2)\sqrt{C + Dt^2}}$. Now, put $C + Dt^2 = u^2$ reduces it into the form $\int \frac{du}{u^2 \pm a^2}$
15.	$\int \frac{ax^2 + bx + c}{(dx + e)\sqrt{fx^2 + gx + h}}\,dx$	Put $ax^2 + bx + c = A_1(dx + e)\frac{d}{dx}(fx^2 + gx + h) + B_1(dx + e) + c_1$, where A_1, B_1, C_1 are constants.
16.	$\int x^m(a + bx^n)^p\,dx$, where m, n, p are rational numbers.	***Case* I** If p is a positive integer, then integrand is expanded by binomial expansion. ***Case* II** If p is negative integer, then substitute $x = t^k$, where k is LCM of denominator of m and n. ***Case* III** If $\frac{m + 1}{n}$ is an integer, then put $a + bx^n = t^\alpha$, where α is denominator of the fraction p. ***Case* IV** If $\frac{m + 1}{n} + p$ is an integer, then put $a + bx^n = t^\alpha x^n$, where α is denominator of the fraction p.
17.	$I_1 = \int \tan^m x \cdot \sec^n x\,dx$, $I_2 = \int \cot^m x \cdot \operatorname{cosec}^n x\,dx$	***Case* I** If n is even positive integer, then put $z = \tan x$ in I_1 and $z = \cot x$ in I_2. ***Case* II** If m is even positive integer, then put $(\sec^2 x - 1)$ in place of $\tan^2 x$ in I_1 and $(\operatorname{cosec}^2 x - 1)$ in place of $\cot^2 x$ in I_2 and substitute $z = \tan x$ in I_1 and $z = \cot x$ in I_2. ***Case* III** If $m = 0$ and n is odd positive integer greater than 1, then method of integration by parts is used.

Integration of Hyperbolic Function

Following are the integration of hyperbolic function:

(i) $\int \sinh x\,dx = \cosh x + C$

(ii) $\int \cosh x\,dx = \sinh x + C$

(iii) $\int \text{sech}^2 x\,dx = \tanh x + C$

(iv) $\int \text{cosech}^2 x\,dx = -\coth x + C$

(v) $\int \text{sech}\ x\,dx = -\text{sech}\ x + C$

(vi) $\int \text{cosech}\ x\,dx = -\text{cosech}\ x + C$

Reduction Formulae

Some important reduction formulae are as follow:

(i) $\int \sin^n x\,dx = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n}\int \sin^{n-2} x\,dx$

(ii) $\int \cos^n x\,dx = \dfrac{\cos^{n-1} x \sin x}{n} + \dfrac{n-1}{n}\int \cos^{n-2} x\,dx$

(iii) $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(iv) $\int \cot^n x\,dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$

(v) $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{(n-2)}{(n-1)}\int \sec^{n-2} x\,dx$

MASTER STROKES

1. Let $f(x)$ be a polynomial satisfying $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. Then, $f(4)$ equals

(a) $\dfrac{5(e^8+1)}{2e^4}$ (b) $\dfrac{5(e^8-1)}{2e^4}$

(c) $\dfrac{2e^4}{5(e^8-1)}$ (d) $\dfrac{2e^4}{5(e^8+1)}$

2. $\int (1 + x - x^{-1})\, e^{x+x^{-1}}\,dx$ equals

(a) $(x+1)e^{x+x^{-1}} + C$

(b) $(x-1)\, e^{x+x^{-1}} + C$

(c) $-xe^{x+x^{-1}} + C$

(d) $xe^{x+x^{-1}} + C$

3. If $I_n = \int (\log x)^n\,dx$, then $I_n + nI_{n-1}$ equals

(a) $(x \log x)^n$ (b) $x(\log x)^n$

(c) $n(\log x)^n$ (d) $(\log x)^{n-1}$

4. $\int \dfrac{\sec^2 x}{(\sec x + \tan x)^{9/2}}\,dx$ equals

(a) $-\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2\right\} + C$

(b) $\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} - \dfrac{1}{7}(\sec x + \tan x)^2\right\} + C$

(c) $-\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2\right\} + C$

(d) $\dfrac{1}{(\sec x + \tan x)^{11/2}}\left\{\dfrac{1}{11} + \dfrac{1}{7}(\sec x + \tan x)^2\right\} + C$

5. If $f(x) = \int \left(\dfrac{x^2 + \sin^2 x}{1 + x^2}\right)\sec^2 x\,dx$ and $f(0) = 0$, then $f(1)$ equals

(a) $1 - \dfrac{\pi}{4}$ (b) $\dfrac{\pi}{4}$

(c) $\tan 1 + \dfrac{\pi}{4}$ (d) $\tan 1 + 1$

6. If $\int f(x)\,dx = \phi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to

(a) $\dfrac{1}{3}\{x^3\phi(x^3) - \int x^2\phi(x^3)\,dx\} + C$

(b) $\dfrac{1}{3}x^3\phi(x^3) - 3\int x^3\phi(x^3)\,dx + C$

(c) $\dfrac{1}{3}x^3\phi(x^3) - \int x^2\phi(x^3)\,dx + C$

(d) $\dfrac{1}{3}\{x^3\phi(x^3) - \int x^3\phi(x^3)\,dx\} + C$

7. If $\int \dfrac{1 - 5\sin^2 x}{\cos^5 x \sin^2 x}\,dx = \dfrac{f(x)}{\cos^5 x} + C$, then $f(x)$ equals

(a) $-\cot x$ (b) $-\text{cosec}\ x$

(c) $\text{cosec}\ x$ (d) $\cot x$

8. For any natural number m, $\int (x^{7m} + x^{2m} + x^m)(2x^{6m} + 7x^m + 14)^{1/m}\,dx\ (x > 0)$ equals

(a) $\dfrac{(7x^{7m} + 2x^{2m} + 14x^m)^{\frac{m+1}{m}}}{14(m+1)} + C$

(b) $\dfrac{(2x^{7m} + 14x^{2m} + 7x^m)^{\frac{m+1}{m}}}{14(m+1)} + C$

(c) $\dfrac{(2x^{7m} + 7x^{2m} + 14x^m)^{\frac{m+1}{m}}}{14(m+1)} + C$

(d) $\dfrac{(7x^{7m} + 2x^{2m} + x^m)^{\frac{m+1}{m}}}{14(m+1)} + C$

9. $\int \dfrac{x}{\sqrt{1 + x^2 + \sqrt{(1+x^2)^3}}}\,dx$ is equal to

(a) $\dfrac{1}{2}\ln(1 + \sqrt{1 + x^2}) + C$ (b) $\dfrac{-2}{3(1 + \sqrt{1 + x^2})^{3/2}} + C$

(c) $2(1 + \sqrt{1 + x^2}) + C$ (d) $2\sqrt{1 + \sqrt{1 + x^2}} + C$

10. $\int e^{x^4}(x + x^3 + 2x^5)e^{x^2}dx$ is equal to

(a) $\frac{1}{2}xe^{x^2}e^{x^4} + C$

(b) $\frac{1}{2}x^2e^{x^4} + C$

(c) $\frac{1}{2}e^{x^2}e^{x^4} + C$

(d) $\frac{1}{2}x^2e^{x^2}e^{x^4} + C$

11. $\int \sin(101x)\sin^{99}x\,dx$ equals

(a) $\frac{1}{100}\sin(100x)\,(\sin x)^{100} + C$

(b) $\frac{1}{100}\cos(100x)\,(\sin x)^{100} + C$

(c) $\frac{1}{100}\cos(100x)\,(\cos x)^{100} + C$

(d) $\frac{1}{100}\sin(100x)\,(\sin x)^{101} + C$

12. If $\int \frac{1 - 7\cos^2 x}{\sin^7 x \cos^2 x}dx = \frac{f(x)}{\sin^7 x} + C$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$

(c) $\tan x$ (d) $\cot x$

13. $\int (x^2 + x)\,(x^{-8} + 2x^{-9})^{1/10}\,dx$ is equal to

(a) $\frac{5}{11}(x^2 + 2x)^{11/10} + C$

(b) $\frac{5}{11}(x + 1)^{11/10} + C$

(c) $\frac{6}{7}(x + 1)^{11/10} + C$

(d) $\frac{11}{5}(x^2 + 2x)^{11/10} + C$

14. If $\int \frac{e^x(2 - x^2)}{(1 - x)\sqrt{1 - x^2}}dx = \mu e^x\left(\frac{1 + x}{1 - x}\right)^{\lambda} + C$, then the value of $2(\lambda + \mu)$ is

(a) 1 (b) 2

(c) 3 (d) 6

15. Let $f(x)$ be a quadratic function such that $f(0) = 1$ and $\int \frac{f(x)}{x^2(x + 1)^3}dx$ is a rational function, then the value of $f'(0)$ is

(a) 0 (b) 2

(c) 3 (d) 5

16. $\int e^x \frac{(x - 1)\,(x - \log x)}{x^2}dx$ is equal to

(a) $e^x\left(\frac{x - \ln x}{x}\right) + C$

(b) $e^x\left(\frac{x - \ln x + 1}{x}\right) + C$

(c) $e^x\left(\frac{x - \ln x}{x^2}\right) + C$

(d) $e^x\left(\frac{x - \ln x - 1}{x}\right) + C$

17. If $\int x^{27}(6x^2 + 5x + 4)\,(x^2 + x + 1)^6\,dx = f(x) + C$, then $f(x)$ is equal to

(a) $\frac{1}{7}(x^6 + x^5 + x^4)^7$ (b) $\frac{1}{7}(6x^5 + 5x^4 + 4x^3)^7$

(c) $\frac{1}{7}(6x^6 + 5x^5 + 4x^4)^7$ (d) $\frac{1}{7}(x^5 + x^4 + x^3)^7$

18. $\int \frac{e^{\tan^{-1}x}}{(1 + x^2)}\left\{(\sec^{-1}\sqrt{1 + x^2})^2 + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)\right\}dx$ is equal to

(a) $e^{\tan^{-1}x}(\tan^{-1}x) + C$

(b) $e^{\tan^{-1}x}(\sec^{-1}\sqrt{1 + x^2})^2 + C$

(c) $\frac{1}{2}e^{\tan^{-1}x}(\tan^{-1}x)^2 + C$

(d) $e^{\tan^{-1}x}(\operatorname{cosec}^{-1}\sqrt{1 + x^2})^2 + C$

19. $\int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2}dx$ is equal to

(a) $\frac{-x}{(x\tan x + 1)} + 2\ln|x\sin x + \cos x| + C$

(b) $\frac{-x^2}{(x\tan x + 1)} + 2\ln|x\sin x + \cos x| + C$

(c) $\frac{-x^2}{(x\tan x + 1)} + 2\ln|x\sin x + \cos x| + C$

(d) None of the above

20. $\int \frac{mx^{m + 2n - 1} - nx^{n - 1}}{x^{2m + 2n} + 2x^{m + n} + 1}dx$ is equal to

(a) $\frac{-x^m}{x^{m+n} + 1} + C$ (b) $\frac{x^n}{x^{m+n} + 1} + C$

(c) $\frac{-x^n}{x^{m+n} + 1} + C$ (d) $\frac{x^m}{x^{m+n} + 1} + C$

21. $\int \frac{dx}{\tan x + \cot x + \sec x + \operatorname{cosec} x}$ is equal to

(a) $\frac{1}{2}(\sin x + \cos x + x) + C$

(b) $\frac{1}{2}(\sin x - \cos x - x) + C$

(c) $\frac{1}{2}(\cos x - x + \sin x) + C$

(d) $\frac{1}{2}(-\cos x + \sin x - 1) + C$

22. $\int \frac{x^2(1 - \ln x)}{(\ln x)^4 - x^4}dx$ is equal to

(a) $\frac{1}{2}\ln\left(\frac{x}{\ln x}\right) - \frac{1}{4}\ln(\ln^2 x - x^2) + C$

(b) $\frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

(c) $\frac{1}{4}\ln\left(\frac{\ln x + x}{\ln x - x}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

(d) $\frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) + \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

To be Continued at Page 65

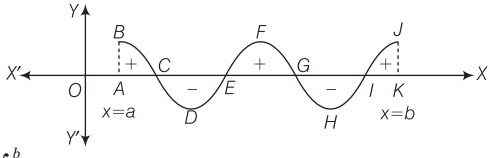
DEFINITE INTEGRATION

Let f be a function of x defined in the closed interval $[a, b]$ and F be another function, such that $F'(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ is called definite integral of the function $f(x)$ over the interval $[a, b]$, a and b are called the limits of integration, a being the lower limit and b being the upper limit. The interval $[a, b]$ is called the interval of integration.

Geometrical Interpretation of Definite Integral

(i) If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$, $x = b$.

(ii) In general, $\int_a^b f(x)\,dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the X-axis and the straight lines $x = a$, $x = b$. The areas above X-axis are taken with plus sign and the areas below X-axis are taken with minus sign



i.e. $\int_a^b f(x)\,dx = \text{ar}\,(ABC) - \text{ar}\,(CDE) + \text{ar}\,(EFG) - \text{ar}\,(GHI) + \text{ar}\,(IJK)$

POINTS TO REMEMBER

- If $f(x)$ is not defined at $x = a$ and $x = b$, but defined in open interval (a, b), then $\int_a^b f(x)\,dx$ can be evaluated.
- In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in the closed interval $[a, b]$.
- If $\int_a^b f(x)\,dx = 0$, then the equation $f(x) = 0$ has atleast one root lying in (a, b) provided f is a continuous function in (a, b).

Properties of Definite Integral

Following are some basic properties of definite integrals:

(i) $\int_a^b f(x)dx = \int_a^b f(t)dt$ [changing variable]

(ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ [interchanging limit]

(iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where c may lie inside the interval $[a, b]$. [splitting limits]

Note This property is useful when function is discontinuous or non-differentiable at $x = c$. This property can be generalised in the form

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx$$

(iv) $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

(v) $\int_a^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

(vi) $\int_0^{2a} f(x)dx = \begin{cases} 0, & \text{if } f(2a - x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(2a - x) = f(x) \end{cases}$

(vii) $\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \end{cases}$

Graphically

$\int_{-a}^{a} f(x)dx = 0$, when $f(x)$ is symmetric about origin (Fig. 1) and

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, when $f(x)$ is symmetric about Y-axis (Fig. 2)

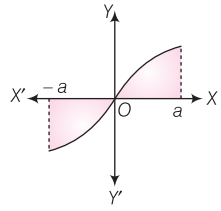


Fig. 1

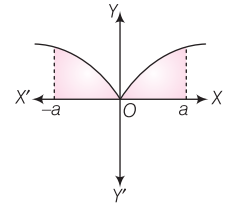


Fig. 2

REMEMBER

- $\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right) = -\frac{\pi}{2} \log (2)$
- $\int_0^{2a} f(x)dx = \int_0^a \{f(a - x) + f(a + x)\}\,dx$
- $\int_a^b f(x)dx = (b - a)\int_0^1 f[(b - a)x + a]dx$

Evaluation of Definite Integral by Substitution

To evaluate integral of the form $\int_a^b f\{g(x)\}\,g'(x)\,dx$, we use following steps :

***Step* I** Put $g(x) = t \Rightarrow g'(x)\,dx = dt$

***Step* II** Find the limits of integration for new variable, i.e. $g(a)$ and $g(b)$. The integral reduces to $\int_{g(a)}^{g(b)} f(t)\,dt$.

***Step* III** Integrate the definite integral obtained in Step II, with the help of suitable formula.

Caution The substitution must be continuous in the interval of integration.

Summation of Series by Definite Integral

Another way of describing $\int_a^b f(x)\,dx$ is that, it is a limiting case of summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$, i.e. $\int_a^b f(x)\,dx = \lim_{n \to \infty} h \sum_{r=1}^{n-1} f(a + rh)$,

where $h = \frac{b-a}{n}$. The converse is also true, i.e. if we have an infinite series of the above form, it can be expressed as definite integral. In order to express the infinite series as definite integral, we use following steps:

***Step* I** Express the given series in the form of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right)$.

***Step* II** Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\lim_{n\to\infty} \Sigma$ by the sign of integration $(\int)$.

***Step* III** The lower and upper limit of integration are limiting values of $\frac{r}{n}$ for the first and the last terms respectively,

i.e. lower limit $= \lim_{n\to\infty} \frac{r_1}{n}$, upper limit $= \lim_{n\to\infty} \frac{r_n}{n}$

where, r_1 and r_n are the least and greatest values of r, respectively.

Partial Integration Formula

If f is continuous and α is monotonic on $[a, b]$, then

$$\int_a^b f(x)\, d\alpha(x) = [f(x)\,\alpha(x)]_a^b - \int_a^b \alpha(x) df(x)$$

or $$\int_a^b f(x)\, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\, df(x)$$

Advanced Properties of Definite Integral

(i) If $f(x)$ is periodic function with period T {i.e. $f(x+T) = f(x)$}. Then, $\int_a^{a+T} f(x)\, dx$ is independent of a.

(a) $\int_0^{nT} f(x)\, dx = n\int_0^T f(x)\, dx,\ n \in I$

(b) $\int_{a+mT}^{a+nT} f(x)\, dx = (n-m)\int_0^T f(x)\, dx;\ m, n \in I$

(c) $\int_{a+mT}^{b+mT} f(x)\, dx = \int_a^b f(x)\, dx,\ m \in I$

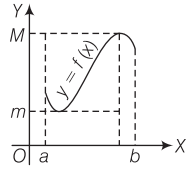
(d) $\int_{nT}^{a+nT} f(x)\, dx = \int_0^a f(x)\, dx$

(ii) If $f(x) \le \phi(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\, dx \le \int_a^b \phi(x)\, dx$

(iii) If at every point x of an interval $[a, b]$, the inequalities $g(x) \le f(x) \le h(x)$ are fulfilled, then $\int_a^b g(x)\, dx \le \int_a^b f(x)\, dx \le \int_a^b h(x)\, dx$.

(iv) $\left|\int_a^b f(x)\, dx\right| \le \int_a^b |f(x)|\, dx$

(v) If m is the least value and M is the greatest value of the function $f(x)$ in the interval $[a, b]$, then $m(b-a) \le \int_a^b f(x)\, dx \le M(b-a)$



(vi) If f is continuous on $[a, b]$, then there exists a number in $[a, b]$ at which $f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx$, is called the mean value of the function $f(x)$ on the interval $[a, b]$.

(vii) If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)\, g(x)\, dx\right| \le \left(\int_a^b f^2(x)\, dx\right)^{1/2} \left(\int_a^b g^2(x)\, dx\right)^{1/2}$$

(viii) Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. Then, for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha)\, dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\, dx$$

(ix) If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\, dt$ is an even function.

(x) If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)\, dt$ is an odd function.

(xi) If $f(t)$ is an even function, then for non-zero a, $\int_a^x f(t)\, dt$ is not necessarily an odd function. If it will be an odd function, then $\int_0^a f(t)\, dt = 0$.

(xii) If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\, dx$ is called an improper integral and is defined as

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

Leibnitz Rule for the Differentiation under Integral Sign

If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and are differentiable at a point $x \in (a, b)$ and $f(t)$ is continuous on $[\phi(a), \phi(b)]$, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\, dt\right] = \frac{d}{dx}\{\psi(x)\}\, f\{\psi(x)\} - \frac{d}{dx}\{\phi(x)\}\, f\{\phi(x)\}$$

Walli's Formula

(i) $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

(ii) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \int_0^{\pi/2} \sin^n x \cos^m x\, dx.$

$$= \begin{cases} \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}\cdot\frac{\pi}{2}, \\ \qquad \text{when both } m, n \text{ are even integers} \\ \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)\cdot(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(1 \text{ or } 2)}, \\ \qquad \text{when either } m \text{ or } n \text{ is odd integer} \end{cases}$$

Gamma Function

The improper integral $\int_0^\infty e^{-x}x^{n-1}dx, n \in I^+$ is defined as gamma function and is denoted by $\Gamma(n)$,

i.e. $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx$, where $n \in Q^+$.

Properties of Gamma Function

(i) $\Gamma(n+1) = n\ \Gamma n = n!$ (ii) $\Gamma 1 = 1$

(iii) $\Gamma 0 = \infty$ (iv) $\left|\frac{1}{2} = \sqrt{\pi}\right.$

(v) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\left|\left(\frac{m+1}{2}\right)\right|\left(\frac{n+1}{2}\right)}{2\left|\left(\frac{m+n+2}{2}\right)\right.}$

POINTS TO REMEMBER

- $\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1$
- $\Gamma m\left|\left(m+\frac{1}{2}\right)\right. = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$
- $\left|\frac{1}{n}\right|\frac{2}{n}\ldots\left|\frac{n-1}{n}\right. = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

Beta Function

The beta function is defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \text{ where } m, n > 0$$

Properties of Beta Function

(i) $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

(ii) $B(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma(m+n)}$, where $m, n > 0$

(iii) $B(m,n) = B(n,m)$, where $m, n > 0$

IMPORTANT SERIES

(i) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (iii) $\sum_{r=1}^{n} r^3 = \left|\frac{n(n+1)}{2}\right|^2$

(iv) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin[\alpha+(n-1)\beta]$

$$= \frac{\sin\frac{n\beta}{2}\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]}{\sin\frac{\beta}{2}}$$

(v) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$

$$= \frac{\sin\frac{n\beta}{2}\cos\left[\alpha+(n-1)\frac{\beta}{2}\right]}{\sin\frac{\beta}{2}}$$

(vi) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots = \frac{\pi^2}{12}$

(vii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots = \frac{\pi^2}{6}$

(viii) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

(ix) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots = \frac{\pi^2}{24}$

(x) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

(xi) $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$ and $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

Some Important Results

(i) (a) $\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx = \frac{a}{2}$

(b) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$

(ii) (a) $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a+b}{a-b}}\right), a > b > 0$

or $\int_0^{\pi/2} \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}-\sqrt{b-a}}{\sqrt{b+a}+\sqrt{b-a}}\right|,$

$0 < a < b$

(b) $\int_0^{\pi/2} \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\right), a > b > 0$

or $\int_0^{\pi/2} \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}}\log\left|\frac{\sqrt{b+a}+\sqrt{b-a}}{\sqrt{b+a}-\sqrt{b-a}}\right|,$

$0 < a < b$

(iii) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x+\cos^n x}dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1+\tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1+\cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x+\cot^n x}dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x+\cot^n x}dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x+\operatorname{cosec}^n x}dx = \frac{\pi}{4}$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x+\operatorname{cosec}^n x}dx, \text{ where } n \in R$$

(iv) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x}+a^{\cos^n x}}dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x}+a^{\cos^n x}}dx = \frac{\pi}{4}$

(v) (a) $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2$

(b) $\int_0^{\pi/2} \log\tan x\,dx = \int_0^{\pi/2} \log\cot x\,dx = 0$

(c) $\int_0^{\pi/2} \log\sec x\,dx = \int_0^{\pi/2} \log\operatorname{cosec} x\,dx = \frac{\pi}{2}\log 2$

(vi) (a) $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

(b) $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$

(c) $\int_0^\infty e^{-ax}x^n dx = \frac{n!}{a^n+1}$

MASTER STROKES

1. The value of $\int_0^3 (x^2+1)\, d[x]$, where $[\cdot]$ denotes the greatest integer function, is

(a) 12 *(b)* 15 *(c)* 17 *(d)* 19

2. If $f(x)$ is an integral function on $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$. Let

$I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta\, f(2\sin 2\theta)d\theta$ and

$I_2 = \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta\, f(2\sin 2\theta)d\theta$, then

(a) $I_1 = 2I_2$ *(b)* $I_1 = 3I_2$

(c) $2I_1 = I_2$ *(d)* None of these

3. If $k \in N$ and $I_R = \int_{-2k\pi}^{2k\pi} |\sin x|\,[\sin x]\, dx$, where $[\cdot]$ denotes the greatest integer function, then $\sum_{k=1}^{100} I_R$ equals

(a) – 10100 *(b)* – 40400 *(c)* – 20200 *(d)* 20200

4. The value of $\int_{t+2\pi}^{t+5\pi/2} \{\sin^{-1}(\cos x) + \cos^{-1}(\cos x)\}\, dx$ is

(a) $\frac{\pi^2}{2}$ *(b)* $\frac{\pi^2}{8}$ *(c)* $\frac{\pi^2}{4}$ *(d)* None of these

5. The value of

$\int_0^{\pi/4} (\tan^n x + \tan^{n-2} x)\, d\left(x - \frac{[x]}{1!} + \frac{[x]^2}{2!} - \frac{[x]^3}{3!} + \ldots\right)$,

where $[x]$ denotes the greatest integer function, is

(a) $\frac{1}{n}$ *(b)* $\frac{1}{n+2}$ *(c)* $\frac{1}{n-1}$ *(d)* $\frac{1}{n-2}$

6. The value of $a\ (>0)$ such that $\int_0^a [\tan^{-1}\sqrt{x}]\, dx = \int_0^a [\cot^{-1}\sqrt{x}]\, dx$, where $[\cdot]$ denotes the greatest integer function, is

(a) $\frac{2(3+\cos 4)}{1-\cos 4}$ *(b)* $\frac{2(3-\cos 4)}{1+\cos 4}$

(c) $\frac{2(3-\cos 4)}{1-\cos 4}$ *(d)* $\frac{2(3+\cos 4)}{1+\cos 4}$

7. If $\alpha, \beta\ (\beta > \alpha)$ are the roots of $f(x) \equiv ax^2 + bx + c = 0$ and $f(x)$ is an even function, then

$\int_\alpha^\beta \frac{e^{f\left(\frac{f(x)}{x-\alpha}\right)}}{e^{f\left(\frac{f(x)}{x-\alpha}\right)} + e^{f\left(\frac{f(x)}{x-\beta}\right)}}\, dx$ equals

(a) $\left|\frac{b}{a}\right|$ *(b)* $\left|\frac{b}{2a}\right|$ *(c)* $\frac{\sqrt{b^2-4ac}}{|2a|}$ *(d)* 0

8. $\frac{\int_{-x}^{x} f(t)\, dt}{\int_0^{2x} f(t+4)\, dt}$ is equal to

(a) $f(0)$ *(b)* 0 *(c)* $\frac{f(x)}{f(0)}$ *(d)* $\frac{f(0)}{f(x)}$

9. $\lim_{x\to 0} \frac{\int_0^x xe^{t^2} dt}{1+x-e^x}$ is equal to

(a) 1 *(b)* – 1 *(c)* 2 *(d)* – 2

10. $\int_0^{[x]/3} \frac{8^x}{2^{[3x]}}\, dx$, where $[\cdot]$ denotes the greatest integer function, is equal to

(a) $\frac{[x]}{\ln 2}$ *(b)* $\frac{[x]}{\ln 4}$ *(c)* $\frac{2[x]}{\ln 2}$ *(d)* $\frac{[x]}{\ln 8}$

11. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g\{x(1-x)\}\, dx$ and

$I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}\, dx$, where g is not an identity function, then the value of I_2 / I_1 is

(a) – 1 *(b)* 1/2 *(c)* 2 *(d)* 1

12. $\int_0^\pi \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\, dx$ equals

(a) $4\sqrt{3} - 4$ *(b)* $4\sqrt{3} - 4 - \frac{\pi}{3}$

(c) $\pi - 4$ *(d)* $\frac{2\pi}{3} - 4 - 4\sqrt{3}$

13. Let $f : \left[\frac{1}{2}, 1\right] \in R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then, the value of $\int_{1/2}^1 f(x)\, dx$ lies in the interval

(a) $(2e-1, 2e)$ *(b)* $(e-1, 2e-1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ *(d)* $\left(0, \frac{e-1}{2}\right)$

14. The maximum value of $f(x) = \int_0^1 t\sin(x+\pi t)\, dt$ is

(a) $\frac{1}{\pi}\sqrt{\pi^2+4}$ *(b)* $\frac{1}{\pi^2}\sqrt{\pi^2+4}$

(c) $\sqrt{\pi^2+4}$ *(d)* $\frac{1}{2\pi^2}\sqrt{\pi^2+4}$

15. If $f(x)$ is a continuous function such that $f(x) > 0$ for all $x > 0$ and $\{f(x)\}^{2020} = 1 + \int_0^x f(t)\, dt$, then the value of $\{f(2020)\}^{2019}$ is

(a) 2019 *(b)* 2020 *(c)* 2021 *(d)* 2018

16. If a function $y = f(x)$ such that $f'(x)$ is continuous function and satisfies $(f(x)^2) = k + \int_0^x [\{f(t)\}^2 + \{f'(t)\}^2]\, dt$, $k \in R^+$, then

(a) $f(x)$ is an increasinig function for all $x \in R$

(b) $f(x)$ is a bounded function

(c) $f(x)$ is neither even nor odd function

(d) If $k = 100$, then $f(0) = 10$

17. The options(s) with the values of a and L that satisfy the equation

$$\frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at)\, dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at)\, dt} = L, \text{ is/are}$$

(a) $a = 2,\ L = \dfrac{e^{4\pi} - 1}{e^{\pi} - 1}$ (b) $a = 2,\ L = \dfrac{e^{4\pi} + 1}{e^{\pi} + 1}$

(c) $a = 4,\ L = \dfrac{e^{4\pi} - 1}{e^{\pi} - 1}$ (d) $a = 4,\ L = \dfrac{e^{4\pi} + 1}{e^{\pi} + 1}$

18. $\int_2^4 \dfrac{\log x^2}{\log x^2 + \log (36 - 12x + x^2)}\, dx$ is equal to

(a) 1 (b) 6 (c) 2 (d) 4

19. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then, the correct expression(s) is/are

(a) $\int_0^{\pi/4} x\ f(x)\, dx = \dfrac{1}{12}$ (b) $\int_0^{\pi/4} x\ f(x)\, dx = 0$

(c) $\int_0^{\pi/4} x\ f(x)\, dx = \dfrac{1}{6}$ (d) $\int_0^{\pi/4} f(x)\, dx = 1$

20. The value of $\dfrac{\left\{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\dfrac{1+x}{1-x}\right) dx\right\}}{\left\{\int_0^{1/2} \cos 2x \cdot \log\left(\dfrac{1+x}{1-x}\right) dx\right\}}$ is

(a) 0 (b) 4 (c) 2 (d) 8

21. Let $f : [0, 2] \in R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})\, dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

(a) $e^2 - 1$ (b) $e^4 - 1$ (c) $e - 1$ (d) e^4

22. $\int_{\pi/4}^{\pi/2} (2\ \text{cosec}\ x)^{17}\, dx$ is equal to

(a) $\int_{\log(1+\sqrt{2})}^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16}\, du$ (b) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17}\, du$

(c) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17}\, du$ (d) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16}\, du$

23. Let $f : R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x. Then, the value of $\int_{-1}^{2} \dfrac{x\, f(x^2)}{2 + f(x+1)}\, dx$ is

(a) $\dfrac{1}{4}$ (b) $\dfrac{1}{2}$ (c) 8 (d) $\dfrac{1}{4}$

24. Let $f : (0, \infty) \to R$ be continuous function such that $F(x) = \int_0^x t\ f(t)\, dt$. If $F(x^2) = x^4 + x^5$, then $\sum_{r=1}^{12} f(r^2)$ is equal to

(a) 216 (b) 219 (c) 222 (d) 225

25. If $f : [0, 1] \to [0, \infty)$ is differentiable function with decreasing first derivative such that $f(0) = 0$ and $f'(x) > 0$, then

(a) $\int_0^1 \dfrac{dx}{f^2(x) + 1} > \dfrac{f(1)}{f'(1)}$ (b) $\int_0^1 \dfrac{dx}{f^2(x) + 1} < \dfrac{f(1)}{f'(1)}$

(c) $\int_0^1 \dfrac{dx}{f^2(x) + 1} \le \dfrac{\tan^{-1}(f(1))}{f'(1)}$ (d) $\int_0^1 \dfrac{dx}{f^2(x) + 1} \le \dfrac{f(1)}{f'(1)}$

26. Let $f : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0, F(3) = -4$ and $F'(x) < 0$ for all $x \in \left(\dfrac{1}{2}, 3\right)$. Let $f(x) = x\, F(x)$ for all $x \in R$. If $\int_1^3 x^2 F'(x)\, dx = -12$ and $\int_1^3 x^3 F''(x)\, dx = 40$, then which of the following is/are correct?

(a) $9f'(3) + f'(1) - 32 = 0$ (b) $\int_1^3 f(x)\, dx = 12$

(c) $9f'(3) - f'(1) + 32 = 0$ (d) $\int_1^3 f(x)\, dx = -12$

27. If $\alpha = \int_0^1 e^{9x + 3\tan^{-1} x} \left(\dfrac{12 + 9x^2}{1 + x^2}\right) dx$, where $\tan^{-1} x$ takes only principal values, then the value of $\left(\log_e |1 + \alpha| - \dfrac{3\pi}{4}\right)$ is

(a) 9 (b) 8
(c) 7 (d) 6

28. If $I_n = \int_0^{\pi} e^x \sin^n x\, dx$, then $\dfrac{I_3}{I_1}$ is equal to

(a) $\dfrac{3}{5}$ (b) $\dfrac{1}{5}$
(c) 1 (d) $\dfrac{2}{5}$

29. If $k = \int_0^1 \dfrac{e^t}{1+t}\, dt$, then $\int_0^1 e^t \log_e (1 + t)\, dt$ is equal to

(a) k (b) $2k$
(c) $e \ln 2 - k$ (d) None of these

30. The value of $\int_{-1}^{1} \dfrac{\log(x + \sqrt{1 + x^2})}{x + \log(x + \sqrt{1 + x^2})} f(x)\, dx$

$- \int_{-1}^{1} \dfrac{\log(x + \sqrt{1 + x^2})}{x + \log(x + \sqrt{1 + x^2})} f(-x)\, dx$ is

(a) 0

(b) $2\int_0^1 \dfrac{\log(x + \sqrt{1 + x^2})}{x + \log(x + \sqrt{1 + x^2})} \{f(x) - f(-x)\}\, dx$

(c) $2f(x)$

(d) None of the above

Answers

1. (c)	**2.** (d)	**3.** (c)	**4.** (c)	**5.** (c)
6. (a)	**7.** (c)	**8.** (d)	**9.** (d)	**10.** (d)
11. (c)	**12.** (b)	**13.** (d)	**14.** (b)	**15.** (b)
16. (a,c)	**17.** (a,c)	**18.** (a)	**19.** (a,b)	**20.** (a)
21. (b)	**22.** (a)	**23.** (a)	**24.** (b)	**25.** (c)
26. (c,d)	**27.** (a)	**28.** (a)	**29.** (c)	**30.** (a)

APPLICATION OF INTEGRALS
(AREA OF BOUNDED REGIONS)

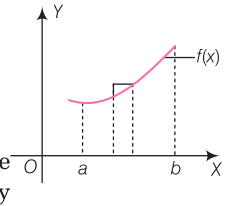
In geometry, we have learnt formulae to calculate areas of various geometrical figures such as triangles, rectangles, trapeziums, circles, etc. By using geometry, we cannot find the area enclosed by many curves. The concept of definite integral can be used to find the areas enclosed by curves.

Area of Bounded Region

Let $y = f(x)$ be a finite and continuous function in the interval $[a, b]$ and

$$A = \int f(x)\,dx = \phi(x) + C$$

$$A = \int_a^b f(x)\,dx = \phi(b) - \phi(a)$$



Similarly, in the same way, it can be shown that the area bounded by any curve $x = g(y)$, Y-axis and two lines $y = c$, $y = d$, is

$$\int_c^d x\,dy \text{ or } \int_c^d g(y)\,dy$$

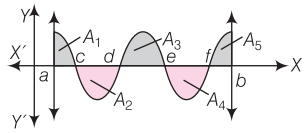
Sign Convention to Find its Area

Following are the important cases to find the sign convention :

***Case* I** In the expression $\int_a^b f(x)\,dx$, if $b > a$ and $f(x) > 0, \forall\, x \in [a, b]$, then this expression will give area enclosed by the curve $f(x)$, X-axis and the lines $x = a$, $x = b$, which is positive. Here, you need not to modify anything.

***Case* II** In the expression $\int_a^b f(x)\,dx$, if $b > a$ and $f(x) < 0, \forall\, x \in [a, b]$, then $\int_a^b f(x)\,dx < 0$ i.e. integrated value is negative, hence area is negative. To make it positive, we take its absolute value.

***Case* III** In the expression $\int_a^b f(x)\,dx$, if $b > a$ and $f(x)$ changes its sign number of times in the interval $[a, b]$, then we need to separate intervals in which $f(x) > 0$ and $f(x) < 0$. Observe the following figure which gives clear thought of it.



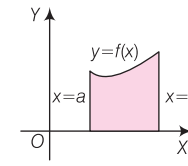
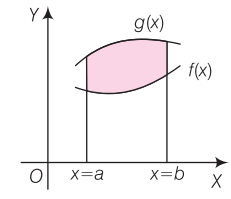
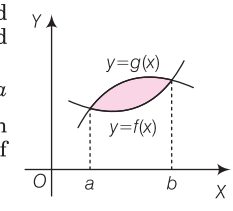
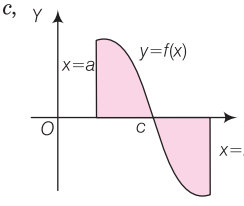
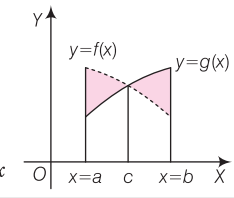
Required area,

$$\int_a^b f(x)dx = A_1 + |A_2| + A_3 + |A_4| + A_5$$

$$\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)\,dx + \left|\int_c^d f(x)\,dx\right| + \int_d^e f(x)\,dx + \left|\int_e^f f(x)\,dx\right| + \int_f^b f(x)\,dx$$

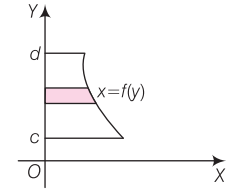
Different Cases of Area of Bounded Region

Following are different cases of area of bounded region :

Cases	Figure
(i) The area bounded by continuous curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$, where $b > a$, is given by $A = \int_a^b f(x)\,dx = \int_a^b y\,dx$	
(ii) The area bounded by the straight lines $x = a$, $x = b$ $(a < b)$ and curves $y = f(x)$, $y = g(x)$ provided $f(x) \le g(x)$ (where, $a \le x \le b$), is given by $A = \int_a^b [g(x) - f(x)]\,dx$	
(iii) When two curves $y = f(x)$ and $y = g(x)$ intersect, then bounded area is given by $A = \int_a^b [g(x) - f(x)]\,dx$, where a and b are the roots of the equation $f(x) = g(x)$ or intersection point of the equation.	
(iv) If the curve crosses the X-axis at c, then the area bounded by the curve $y = f(x)$ and the straight lines $x = a$, $x = b$, where $b > a$, is given by $A = \int_a^c f(x)\,dx + \left\lvert\int_c^b f(x)\,dx\right\rvert$ $= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$	
(v) The area bounded by $y = f(x)$ and $y = g(x)$ (where, $a \le x \le b$) when they intersect at $x = c \in (a, b)$, is given by $A = \int_a^b \lvert f(x) - g(x)\rvert\,dx$ $= \int_a^c [f(x) - g(x)]\,dx + \int_c^b [g(x) - f(x)]\,dx$	

Note The area bounded by a cartesian curve $x = f(y)$, Y-axis and the straight lines $y = c$, $y = d$, is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$



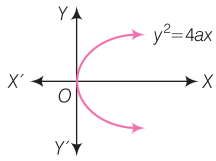
Curve Tracing

For the evaluation of area bounded by the curves, we should have very good understanding of tracing of curve. Following are important steps to be followed to trace the curve :

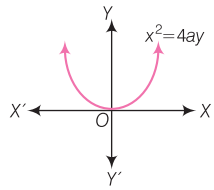
(i) Find the set of permissible values of x (i.e. domain).

(ii) Check the symmetry of the curve. This symmetry can be judged as follows :

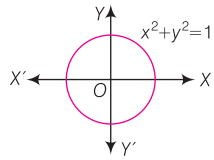
(a) If all the powers of y are even, then the curve is symmetrical about X-axis.



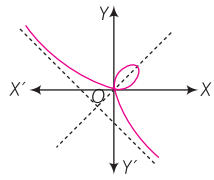
(b) If all the powers of x are even, then the curve is symmetrical about Y-axis.



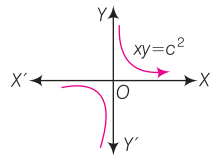
(c) If x and y both have even power, then curve is symmetrical along X-axis as well as Y-axis.



(d) If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.



(e) If on interchanging the sign of x and y and there is no change in equation or equation is unaltered, then the equation is symmetric in opposite quadrants.



(iii) Find the points, where the curve crosses the X-axis and also the Y-axis.

(iv) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve, where you have horizontal tangents.

(v) Find the period of the curve, if it is periodic.

(vi) Find the asymptote(s) of the curve, if any.

Asymptote is a line, whose distance from curve tends to zero as point on the curve moves towards infinity. The following cases of asymptote are as follow:

(a) If $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$, then $x = a$ is asymptote of $y = f(x)$.

(b) If $\lim_{x \to \infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$, then $y = k$ is asymptote of $y = f(x)$.

(c) If $\lim_{x \to \infty} \frac{f(x)}{x} = m_1$ and $\lim_{x \to \infty} [f(x) - m_1 x] = c_1$, then $y = m_1 x + c_1$ is an asymptote inclined to right.

(d) If $\lim_{x \to -\infty} \frac{f(x)}{x} = m_2$ and $\lim_{x \to -\infty} [f(x) - m_2 x] = c_2$, then $y = m_2 x + c_2$ is an asymptote inclined to left.

Important Transformations

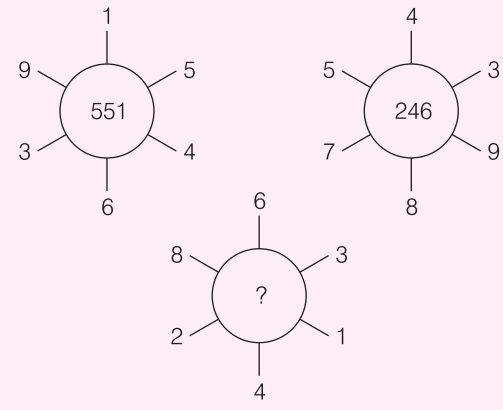
Following are important transformations of the functions:

	Original functions	Transformed functions	Process to transform original functions
(i)	$y = f(x)$	$-f(x)$	Flip about the X-axis.
(ii)	$y = f(x)$	$y = f(-x)$	Flip about the Y-axis.
(iii)	$y = f(x)$	$y = f(x) + k$	Shift the graph k units upwards or downwards depending on whether k is positive or negative.
(iv)	$y = f(x)$	$y = f(x + k)$	Shift left or shift right the graph by k units depending on whether k is positive or negative.
(v)	$y = f(x)$	$y = kf(x)$	Stretch or compress the graph along Y-axis k units depending on $\|k\| > 1$ or $\|k\| < 1$. If k is negative, also flip the curve about the X-axis.
(vi)	$y = f(x)$	$y = f(kx)$	Stretch or compress the graph along X-axis k units depending on $\|k\| > 1$ or $\|k\| < 1$. If k is negative, also flip it along the Y-axis.
(vii)	$y = f(x)$	$y = \|f(x)\|$	Portion of the graph $y = f(x)$ uneffected, if it is positive, take the mirror image of the negative part with respect to X-axis.
(viii)	$y = f(x)$	$y = f(\|x\|)$	Discard the left side of the graph (which lies left of Y-axis) and take mirror image of right side of the graph with respect to Y-axis.
(ix)	$y = f(x)$	$\|y\| = f(x)$	Discard the graph, which is below the X-axis and take the mirror image of the remaining graph with respect to X-axis.

POINTS TO REMEMBER

- The area of the region bounded by $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16ab}{3}$ sq units.
- The area of the region bounded by $y^2 = 4ax$ and $y = mx$ is $\frac{8a^2}{3m^3}$ sq units.
- The area of the region bounded by $y^2 = 4ax$ and its latusrectum is $\frac{8a^2}{3}$ sq units.
- The area of the region bounded by one arc of $\sin ax$ or $\cos ax$ and X-axis is $\frac{2}{a}$ sq units.
- Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab sq units.
- Area bounded by $y = \log_e x$, $y = 0$ and $x = 0$ is 1 sq unit.

Use Your Brain!!!



MASTER STROKES

1. The area enclosed between the curve $y = \log(x + e)$ and the coordinate axes, is
(a) 4 sq units (b) 3 sq units
(c) 2 sq units (d) 1 sq unit

2. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ equals
(a) 1 : 1 : 1 (b) 2 : 1 : 2 (c) 1 : 2 : 3 (d) 1 : 3 : 2

3. The area enclosed between the curves $y^2 = x$ and $y = |x|$, is
(a) $\frac{1}{6}$ sq unit (b) $\frac{1}{3}$ sq unit (c) $\frac{2}{3}$ sq unit (d) 1 sq unit

4. The area of the region bounded by the parabola $(y - 2)^2 = (x - 1)$, the tangent to the parabola at the point (2, 3) and the X-axis, is
(a) 3 sq units (b) 6 sq units
(c) 9 sq units (d) 12 sq units

5. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$, is
(a) $(4\sqrt{2} - 1)$ sq units (b) $(4\sqrt{2} + 1)$ sq units
(c) $(4\sqrt{2} - 2)$ sq units (d) $(4\sqrt{2} + 2)$ sq units

6. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)\,dx$ and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the X-axis. Then,
(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$
(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

7. Let the straight line $x = b$ divides the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \le x \le b)$ and $R_2 (b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals
(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

8. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X-axis, is
(a) $\frac{1}{2}$ sq unit (b) 1 sq unit (c) $\frac{3}{2}$ sq units (d) $\frac{5}{2}$ sq units

9. The area of the region bounded by the curve $y = x^3$ and the lines $y = 8$, $x = 0$, is
(a) 16 sq units (b) 8 sq units
(c) 10 sq units (d) 12 sq units

10. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, which one of the following is incorrect?
(a) $S \ge \frac{1}{e}$ (b) $S \ge 1 - \frac{1}{e}$
(c) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

11. The area bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant, is
(a) 9 sq units (b) 36 sq units
(c) 18 sq units (d) 27/4 sq units

12. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$, is
(a) $4(\sqrt{2} - 1)$ sq units (b) $2\sqrt{2}(\sqrt{2} - 1)$ sq units
(c) $2(\sqrt{2} + 1)$ sq units (d) $2\sqrt{2}(\sqrt{2} + 1)$ sq units

13. The area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$ in two parts. The area of the larger part is

(a) $\frac{245}{4}$ sq units (b) $\frac{147}{4}$ sq units

(c) $\frac{45}{4}$ sq units (d) $\frac{137}{4}$ sq units

14. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$, is

(a) $\left(\frac{\pi}{2} - \frac{2}{3}\right)$ sq units (b) $\left(\frac{\pi}{2} + \frac{2}{3}\right)$ sq units

(c) $\left(\frac{\pi}{2} + \frac{4}{3}\right)$ sq units (d) $\left(\frac{\pi}{2} - \frac{4}{3}\right)$ sq units

15. The area of the region bounded by $y^2 = 2x$ and $y = 4x - 1$, is

(a) $\frac{15}{64}$ sq unit (b) $\frac{9}{32}$ sq unit

(c) $\frac{7}{32}$ sq unit (d) $\frac{5}{64}$ sq unit

16. If $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, where $\alpha \in \{0, 1\}$. Then, the value of $F(\alpha) + \frac{8\sqrt{2}}{3}$ when $\alpha = 0$, is

(a) 4 (b) 5 (c) 6 (d) 9

17. Let $F(x) = \int_x^{x^2 + \pi/6} 2\cos^2 t \, dt$ for all $x \in R$ and $F : [0, 1/2] \to [0, \infty)$ be continuous function. For $a \in [0, 1/2]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ equals

(a) 1 (b) 2 (c) 3 (d) 6

18. The area of the region bounded by the curve $y = \frac{x + 1}{x^2 + 1}$ and the line $y = 1$, is

(a) $\left(1 - \frac{1}{2}\ln 2 + \frac{\pi}{4}\right)$ sq units (b) $\left(\ln 2 - \frac{\pi}{4} + 1\right)$ sq units

(c) $\left(\frac{1}{2}\ln 2 + \frac{\pi}{4} - 1\right)$ sq units (d) $\left(\ln 2 - \frac{\pi}{2} + 1\right)$ sq units

19. The graphs of $f(x) = x^2$ and $g(x) = cx^3$ intersect at two points. If the area of the region bounded between $f(x)$ and $g(x)$ over the interval $[0, 1/c]$ is equal to $\frac{2}{3}$, then the value of $\frac{1}{c} + \frac{1}{c^2}$ is

(a) 20 (b) 2 (c) 6 (d) 12

20. The area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$ which lies to the right of the line $x = 1$, is

(a) $\left(\frac{12 - 20\sqrt{3}}{2}\right)$ sq units (b) $\left(\frac{20 - \sqrt{2}}{3}\right)$ sq units

(c) $\left(\frac{20 - 12\sqrt{2}}{3}\right)$ sq units (d) $\left(\frac{12 - 20\sqrt{2}}{3}\right)$ sq units

Answers

1. (d)	2. (a)	3. (b)	4. (c)	5. (c)
6. (c)	7. (b)	8. (c)	9. (d)	10. (c)
11. (a)	12. (b)	13. (b)	14. (c)	15. (b)
16. (c)	17. (c)	18. (c)	19. (c)	20. (c)

Continued from Page 56

23. The value of $\int \frac{x\cos x + 1}{\sqrt{2x^3 e^{\sin x} + x^2}}\, dx$ is

(a) $\ln\left|\frac{\sqrt{2xe^{\sin x} + 1} - 1}{\sqrt{2xe^{\sin x} + 1} + 1}\right| + C$

(b) $\ln\left|\frac{\sqrt{2xe^{\sin x} - 1} - 1}{\sqrt{2xe^{\sin x} - 1} + 1}\right| + C$

(c) $\ln\left|\frac{\sqrt{2xe^{\sin x} - 1} + 1}{\sqrt{2xe^{\sin x} - 1} - 1}\right| + C$

(d) $\ln\left|\frac{\sqrt{2xe^{\sin x} + 1} + 1}{\sqrt{2xe^{\sin x} - 1} + 1}\right| + C$

24. Let $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n > 2$ and $g(x) = \underbrace{fofofo \ldots f(x)}_{n \text{ times}}$. Then, $\int x^{n-2} g(x)\, dx$ equals

(a) $\frac{1}{n(n-1)}(1 + nx^n)^{1 - \frac{1}{n}} + C$

(b) $\frac{1}{n-1}(1 + nx^n)^{1 - \frac{1}{n}} + C$

(c) $\frac{1}{n(n-1)}(1 + nx^n)^{1 + \frac{1}{n}} + C$

(d) $\frac{1}{n-1}(1 + nx^n)^{1 + \frac{1}{n}} + C$

25. $\int \frac{x^2 - 1}{x\sqrt{(x^2 + \alpha x + 1)(x^2 + \beta x + 1)}}\, dx$ is equal to

(a) $\log\left\{\frac{\sqrt{x^2 + \alpha x + 1} + \sqrt{x^2 + \beta x + 1}}{\sqrt{x}}\right\} + C$

(b) $2\log\left\{\frac{\sqrt{x^2 + \alpha x + 1} - \sqrt{x^2 + \beta x + 1}}{\sqrt{x}}\right\} + C$

(c) $\log\{\sqrt{x^2 + \alpha x + 1} - \sqrt{x^2 + \beta x + 1}\} + C$

(d) None of the above

Answers

1. (b)	2. (d)	3. (b)	4. (c)	5. (c)
6. (c)	7. (d)	8. (c)	9. (d)	10. (d)
11. (a)	12. (c)	13. (a)	14. (c)	15. (c)
16. (d)	17. (a)	18. (b)	19. (b)	20. (c)
21. (d)	22. (b)	23. (a)	24. (a)	25. (a)

DIFFERENTIAL EQUATION

A differential equation is a mathematical equation that relates some function with its derivative.

Differential Equation

An equation involving independent variable(s), dependent variable(s) and derivative(s) of dependent variable(s) with respect to independent variable(s) is called a differential equation.

e.g. $2x\frac{dy}{dx} - 3y = 5$ and $4\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

are differential equations.

Note $x^2 + 2xy + y^2 = 0$ is not a differential equation.

Types of Differential Equation

There are two types of differential equation, which are given below:

(i) **Ordinary differential equation** A differential equation involving derivatives with respect to a single independent variable, is called an ordinary differential equation.

(ii) **Partial differential equation** A differential equation involving derivatives with respect to more than one independent variables, is called a partial differential equation.

But here we will only discuss ordinary differential equation.

Order of Differential Equation

In a differential equation, the order of highest derivative occurring in it is called the order of differential equation.

Degree of Differential Equation

When an equation is a polynomial in derivative(s), the power of highest order derivative is known as degree of differential equation.

e.g. To find degree of differential equation

$\sqrt{\frac{d^3y}{dx^3}} = \sqrt[4]{\left(\frac{dy}{dx} + 5\right)}$, we convert it in polynomial form, by removing its radicals, i.e. write given equation as

$$\left(\sqrt{\frac{d^3y}{dx^3}}\right)^4 = \left(\sqrt[4]{\frac{dy}{dx} + 5}\right)^4$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = \frac{dy}{dx} + 5$$

Hence, given differential equation has order 3 (order of highest derivative) and degree 2 (power of highest order derivative).

Formation of Differential Equation

Consider an equation $f(x, y, c_1, c_2, \ldots, c_n)$ containing n independent arbitrary constants $c_1, c_2, \ldots, c_n$ besides the variables x and y.

The formation of a differential equation representing given equation, means to form a differential equation by eliminating arbitrary constant from the given equation. For this, we differentiate $f(x, y, c_1, c_2, \ldots, c_n)$ n times, we shall get n equations.

Now, among these n equations and the given equation, in all $(n + 1)$ equations, if n arbitrary constants $c_1, c_2, c_3, \ldots, c_n$ are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

Working Rule

Step I Obtained the number of arbitrary constant present in given equation.

Step II Differentiate the given equation n times with respect to x, where n is number of arbitrary constant present.

Step III Eliminate arbitrary constant using $(n + 1)$ equations.

Solution of Differential Equation

Any relation between dependent and independent variable, which when substituted in the differential equation reduce it to in an identity, is called a solution of differential equation. A solution of differential equation does not contain the derivative of the dependent variable with respect to independent variable.

Solutions of differential equation are of two types:

(i) **General solution** The solution of an ordinary differential equation of nth order, which contains n arbitrary constant, is called the general solution.

Let equation involving the variables x, y and n independent arbitrary constant be $f(x, y, c_1, c_2, \ldots, c_n) = 0$ and differential equation obtained from it, is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}\right) = 0,$$

then $f(x, y, c_1, \ldots, c_n)$ is called general solution.

(ii) **Particular solution** Any solution obtained from the general solution of differential equation by giving particular values to the arbitrary constant, is called particular solution.

Various Methods to Solve First Order and First Degree Differential Equation

Here, we are giving various methods to solve first order and first degree differential equation.

Variable Separable Type

If a first order first degree equation can be expressed in such a manner that coefficient of dx is only a function of x, say $f(x)$ and coefficient of dy is only function of y, say $g(y)$, i.e. written in the form $f(x)\,dx + g(y)\,dy = 0$, to solve such differential equation, we integrate both sides,

i.e. $\int f(x)\,dx + \int g(y)\,dy = C$

Reducible to Variable Separable Form

Differential equation of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by substituting $ax + by + c = z$.

$$\Rightarrow \qquad a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$$

Now, given equation becomes

$$\frac{\frac{dz}{dx} - a}{b} = f(z) \quad \Rightarrow \quad \frac{dz}{dx} = b\,f(z) + a$$

$$\Rightarrow \qquad \frac{dz}{b\,f(z) + a} = dx$$

On integrating the above equation, we get

$$\int \frac{dz}{b\,f(z) + a} = x + C,$$

where $z = ax + by + c$

which represents the general solution of the differential equation.

SPEED BOOSTER

If we can write the differential equation in the form $f[f_1(x, y)d\{f_1(x, y)\}] + \phi[f_2(x, y)d\{f_2(x, y)\}] + \ldots = 0$, then each term can be easily integrated separately. For this, following results must be memorised:

- $d(x \pm y) = dx \pm dy$
- $d(xy) = x\,dy + y\,dx$
- $x\,dx + y\,dy = \frac{1}{2}d(x^2 + y^2)$
- $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{x dy - y dx}{x^2}$
- $d\left(\frac{x^2}{y}\right) = \frac{2xy\,dx - x^2 dy}{y^2}$
- $d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2 dx}{x^2}$
- $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2 + y^2}$
- $d\left\{\ln\left(\frac{x}{y}\right)\right\} = \frac{y\,dx - x\,dy}{xy}$
- $d\left\{\ln\left(\frac{y}{x}\right)\right\} = \frac{x\,dy - y\,dx}{xy}$
- $d\{\ln(xy)\} = \frac{x\,dy + y\,dx}{xy}$
- $d\{\ln(x + y)\} = \frac{dx + dy}{x + y}$
- $d\left\{\frac{1}{2}\ln(x^2 + y^2)\right\} = \frac{x\,dx + y\,dy}{x^2 + y^2}$
- $d\left(\frac{-1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$
- $d\left(\frac{e^y}{x}\right) = \frac{xe^y dy - e^y dx}{x^2}$
- $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$
- $d(x^m y^n) = x^{m-1}y^{n-1}(my\,dx + nx\,dy)$
- $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{[f(x,y)]^n}$

Homogeneous Differential Equation

A function $f(x, y)$ is called homogeneous function of degree n, if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

e.g. $f(x, y) = x^3y - y^3x$ is homogeneous function of degree 4, since $f(\lambda x, \lambda y) = \lambda^3x^3\,\lambda y - \lambda^3y^3\,\lambda x$

$$f(\lambda x, \lambda y) = \lambda^4(x^3y - y^3x) = \lambda^4 f(x, y)$$

A differential equation of the form $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a homogeneous function of degree zero, is called a homogeneous differential equation. Such equation is solved by substituting $y = vx$ or $x = yv$ according as $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = G\left(\frac{x}{y}\right)$ and then separate the variables.

Reducible to Homogeneous Form

Equation of the form $\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + f}\,(ae \neq bd)$ can be reduced to a homogeneous form by changing the variables x, y to X, Y by the equations $x = X + h, y = Y + k$, where h, k are constants and are to be chosen such that the given equation becomes homogeneous.

On substituting $x = X + h$ and $y = Y + k$, we get

$$\frac{dy}{dx} = \frac{d(Y + k)}{dx} = \frac{dY}{dx} = \frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

So, given equation becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{dX + eY + (dh + ek + f)}$$

Now, h and k are to be chosen such that $ah + bk + c = 0$ and $dh + ek + f = 0$. This gives $h = \frac{bf - ec}{ae - bd}$, $k = \frac{cd - af}{ae - bd}$

Thus, given equation reduced to $\frac{dY}{dX} = \frac{aX + bY}{dX + eY}$

which can be solved by putting $Y = vX$.

Linear Differential Equation

Differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, is called linear differential equation.

Working Rule

Step I Write the given equation in the form $\frac{dy}{dx} + Py = Q$ and identify P and Q.

Step II Find integrating factor $\text{IF} = e^{\int P\,dx}$.

Step III The required solution is given by $y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$

Note Sometimes, a linear differential equation can be put in the form $\frac{dx}{dy} + Rx = S$, where R and S are functions of y alone, then its solution will be $x \cdot e^{\int R dy} = \int S \cdot e^{\int R dy} dy$.

Bernoulli's Differential Equation

Sometimes a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution.

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ $(n \neq 0, 1)$, where P and Q are functions of x only, is known as Bernoulli's differential equation (for $n = 0$, the equation is linear and for $n = 1$, the equation is in variable separable form).

It is easy to reduce the above equation into linear form as below:

Dividing both sides by y^n, we get $\frac{1}{y^n} \cdot \frac{dy}{dx} + \frac{1}{y^{n-1}} \cdot P = Q$

On putting $y^{1-n} = z$ and hence, $(1-n)\, y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$ the equation becomes $\frac{dz}{dx} + (1-n)Pz = (1-n)Q$ which is linear in z.

Here, $\text{IF} = e^{\int (1-n)\, P\, dx}$ and the solution is

$$z \cdot e^{\int (1-n)P\,dx} = \int \{(1-n) \cdot Q \cdot e^{(1-n)Pdx}\} dx + C$$

REMEMBER

- Another type of equation which is reducible to the linear form is $f'(y)\frac{dy}{dx} + P(x) \cdot f(y) = Q(x)$.
- An equation of this type can be easily reduced to linear form by taking $z = f(y)$.

Application of Differential Equation

Following are important applications of differential equation:

(i) **Growth and decay problem** Let N (t) denotes the amount of substance (or population) that is either growing or decaying. If we assume that $\frac{dN}{dt}$, the time rate of change of this amount of substance, is proportional to amount of substance present, then $\frac{dN}{dt} = kN$ or $\frac{dN}{dt} - kN = 0$, where k is the constant of proportionality.

(ii) **Temperature problem** Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of temperature of a body is proportional to the temperature difference between the body and its surrounding medium.

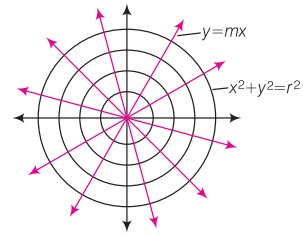
Let T and T_m respectively denote the temperatures of the body and medium, then time rate of change of temperature of the body is $\frac{dT}{dt}$ and Newton's law of cooling can be formulated as $\frac{dT}{dt} = -k\,(T - T_m)$ or $\frac{dT}{dt} + kT = kT_m$, where k is a positive constant of proportionality.

If $k > 0$ and $\frac{dT}{dt} < 0 \Rightarrow$ Cooling process

If $k > 0$ and $\frac{dT}{dt} > 0 \Rightarrow$ Heating process

Orthogonal Trajectories

Any curve, which cuts every member of given family of curves at right angle, is called an orthogonal trajectory of the family. e.g. Each straight line passing through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = r^2$.



Working Rule

Following steps are to be followed to find the orthogonal trajectory of given family of curves:

Step I Let $f(x, y, c) = 0$ be the given equation, where c is an arbitrary parameter.

Step II Differentiate given equation with respect to x and eliminate c.

Step III Substitute $\left(\frac{-dx}{dy}\right)$ for $\left(\frac{-dy}{dx}\right)$ in the equation obtained in Step II.

Step IV Solve the differential equation obtained in Step III, to get required orthogonal trajectories.

Do You Know!!!

Among all shapes with the same perimeter, a circle has the largest area while among all shapes with the same area circle has the shortest perimeter.

MASTER STROKES

1. The solution of the differential equation $x\left(\frac{dy}{dx}\right)^2 + 2\sqrt{xy}\frac{dy}{dx} + y = 0$, is

(a) $x + y = C$ (b) $\sqrt{x} - \sqrt{y} = C$

(c) $x^2 + y^2 = C^2$ (d) $\sqrt{x} + \sqrt{y} = \sqrt{C}$

2. The orthogonal trajectory of the family of curves $a^{n-1}y = x^n$ (a is the arbitrary constant) is given by

(a) $x^n + n^2y = \text{constant}$ (b) $ny^2 + x^2 = \text{constant}$

(c) $n^2x + y^n = \text{constant}$ (d) $n^2x - y^n = \text{constant}$

3. If $\phi(x)$ is a differentiable function, then the solution of the differential equation $dy + \{y\,\phi'(x) - \phi(x)\,\phi'(x)\}\,dx = 0$, is

(a) $y = \{\phi(x) - 1\} + Ce^{-\phi(x)}$

(b) $y\,\phi(x) = \{\phi(x)\}^2 + C$

(c) $ye^{\phi(x)} = \phi(x)\,e^{\phi(x)} + C$

(d) $y - \phi(x) = \phi(x)e^{-\phi(x)}$

4. The solution of the differential equation $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)dy = 0$, is given by

(a) $2\log|x| - \log|y| - \frac{1}{xy} = C$

(b) $2\log|y| - \log|x| - \frac{1}{xy} = C$

(c) $2\log|x| + \log|y| + \frac{1}{xy} = C$

(d) $2\log|y| + \log|x| + \frac{1}{xy} = C$

5. If $y(t)$ is a solution of $(1 + t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to

(a) $-\frac{1}{2}$ (b) $e + \frac{1}{2}$

(c) $e - \frac{1}{2}$ (d) $\frac{1}{2}$

6. The solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$, is

(a) $xe^{2\tan^{-1}y} = e^{\tan^{-1}y} + C$ (b) $(x - 2) = Ce^{\tan^{-1}y}$

(c) $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + C$ (d) $xe^{\tan^{-1}y} = \tan^{-1}y + C$

7. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point, the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x + 1)^2$ (b) $(x - 1)^3$ (c) $(x + 1)^3$ (d) $(x - 1)^2$

8. The differential equation which represents the family of curves $y = C_1e^{C_2x}$, where C_1 and C_2 are arbitrary constants, is

(a) $y' = y^2$ (b) $y'' = y'y$

(c) $yy'' = y'$ (d) $yy'' = (y')^2$

9. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants, is of

(a) second order and second degree

(b) first order and second degree

(c) first order and first degree

(d) second order and first degree

10. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(a) $\log\left(\frac{x}{y}\right) = Cy$ (b) $\log\left(\frac{y}{x}\right) = Cx$

(c) $x\log\left(\frac{y}{x}\right) = Cy$ (d) $y\log\left(\frac{x}{y}\right) = Cx$

11. If $x\,dy = y(dx + y\,dy)$, $y(1) = 1$ and $x > 0$, then $y(-3)$ equals

(a) 3 (b) 2 (c) 1 (d) 0

12. Tangent is drawn at any point P of a curve which passes through (1, 1) cutting X-axis and Y-axis at A and B respectively. If $AP : PB = 3 : 1$, then

(a) the differential equation of the curve is $3x\frac{dy}{dx} + y = 0$ and the curve passes through (1/8, 2)

(b) the differential equation of the curve is $3x\frac{dy}{dx} - y = 0$ and the curve passes through (1/8, – 2)

(c) the curve is passing through (– 1 / 8, – 2)

(d) the normal at (1, 1) is $x + 3y = 4$

13. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(t) = 1$ and $\lim_{t\to x}\frac{t^2f(x) - x^2f(t)}{t - x} = 1$ for each $x > 0$. Then, $f(x)$ is

(a) $\frac{1}{3x} + \frac{2}{3}x^2$ (b) $-\frac{1}{3x} + \frac{4}{3}x^2$

(c) $-\frac{1}{x} + \frac{2}{x^2}$ (d) $\frac{1}{x}$

14. Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x\sqrt{1 - \{f'(t)\}^2}\,dt = \int_0^x f(t)\,dt$, $0 \le x \le 1$ and $f(0) = 0$, then

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$ (b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

15. The interval contained in the domain of definition of non-zero solutions of the differential equation $(x - 3)^2y' + y = 0$, is

(a) $(-\pi/2, \pi/2)$ (b) $(0, \pi)$

(c) $(0, 2\pi)$ (d) $(-\pi, \pi)$

16. The solution of the differential equation $\cos x\, dy = y(\sin x - y)dx$, $0 < x < \frac{\pi}{2}$ is

(a) $y\tan x = \sec x + C$ (b) $\tan x = (\sec x + C)y$

(c) $\sec x = (\tan x + C)y$ (d) $y \sec x = \tan x + C$

17. Let f be a real valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the curve of the abscissa of P, then the value of $f(-3)$ is

(a) 3 (b) 6 (c) 9 (d) 0

18. Let $y'(x) + y(x)g'(x) = g(x)\, g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{d}{dx}(f(x))$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is

(a) 0 (b) 1 (c) -1 (d) 2

19. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{d}{dt}(p(t)) = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(a) $600 - 500e^{t/2}$ (b) $400 - 300\, e^{-t/2}$

(c) $400 - 300e^{t/2}$ (d) $300 - 200\, e^{-t/2}$

20. A curve passes through the point $(1, \pi/6)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. Then, the equation of the curve is

(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

21. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x \sec x$ and $y(0) = 0$, then

(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$, $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

(b) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4\sqrt{2}}$, $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$, $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{\pi^2}{3\sqrt{3}}$

(d) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{4\sqrt{2}}$, $y'\left(\frac{\pi}{3}\right) = \frac{\pi^2}{18}$

22. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\, dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is

(a) 3 (b) 4 (c) 5 (d) 6

23. Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$. Then, $y(e)$ equals

(a) 2 (b) $2e$ (c) e (d) 0

24. The sum of the squares of the perpendicular drawn from the points $(0, 1)$ and $(0, -1)$ to any tangent to a curve is 2. The equation of the curve is

(a) $2y = C(x + 2)$ (b) $y = C(x \pm 1)$

(c) $y = C(x + 2)$ (d) $y = C(x \pm 2)$

25. If the solution of the equation $(2 + x)\, dy - y\, dx = 0$ represents a curve passing through a fixed point P, then the area of equilateral triangle with P as one vertex and $x + y = 0$ as its one side is

(a) $2\sqrt{3}$ sq units (b) $\sqrt{3}$ sq units

(c) $\frac{2}{\sqrt{3}}$ sq units (d) $\frac{4}{\sqrt{3}}$ sq units

26. If $\frac{dx}{dy} = (e^y - x)$, where $y(0) = 0$, then y is expressed explicitly as

(a) $\frac{1}{2}\ln(1 + x^2)$ (b) $\ln(1 + x^2)$

(c) $\ln(x + \sqrt{1 + x^2})$ (d) $\ln(x + \sqrt{1 - x^2})$

27. If the function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{x}{x^2 - 1}y = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)\, dx$ is equal to

(a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

28. Consider the differential equation

$$y^2 dx + \left(x - \frac{1}{y}\right)dy = 0.$$

If $y(1) = 1$, then x is given by

(a) $3 - \frac{1}{y} + \frac{e^{1/y}}{e}$ (b) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$

(c) $1 - \frac{1}{y} + \frac{e^{1/y}}{e}$ (d) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$

29. Consider the differential equation

$$y\, dx - (x + y^2)\, dy = 0.$$

If for $y = 1$, x takes value 1, then the value of x, when $y = 4$, is

(a) 64 (b) 9

(c) 16 (d) 36

30. If $\frac{dy}{dx} = y + 3$ and $y(0) = 2$, then $y(\ln 2)$ is equal to

(a) 7 (b) 5 (c) 13 (d) -2

Answers

1. (d)	2. (b)	3. (a)	4. (a)	5. (a)
6. (c)	7. (b)	8. (d)	9. (d)	10. (b)
11. (a)	12. (a)	13. (a)	14. (c)	15. (a)
16. (c)	17. (c)	18. (a)	19. (c)	20. (a)
21. (a)	22. (d)	23. (a)	24. (b)	25. (c)
26. (c)	27. (b)	28. (b)	29. (c)	30. (a)

Learn Through Problems

LIMIT, CONTINUITY AND DIFFERENTIABILITY

Section targetted to provide stepwise learning & practice through selective problems

1 $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^2\tan x}$ equals

(a) 1 (b) 1/2

(c) 1/3 (d) None of these

(c) ***Step I*** Using direct substitution, check whether the limit is indeterminate form or not.

We have, $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^2\tan x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Step II Use L' Hospital's rule. Differentiate both numerator and denominator separately.

Given limit $= \lim\limits_{x\to 0}\dfrac{\sec^2 x - 1}{2x\tan x + x^2\sec^2 x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

Step III Again use direct substitution to check whether the limit is indeterminate form or not. If yes, use L' Hospital's rule again untill we get the value of limit.

Given limit $= \lim\limits_{x\to 0}\dfrac{2\sec^2 x\tan x}{2\tan x + 4x\sec^2 x + 2x^2\sec^2 x\tan x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to 0}\dfrac{2\sec^4 x + 4\sec^2 x\tan^2 x}{6\sec^2 x + 12x\sec^2 x\tan x + 2x^2\sec^4 x + 4x^2\sec^2 x\tan^2 x}$

$= \dfrac{2}{6} = \dfrac{1}{3}$

1. If $\lim\limits_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then $a + b + c$ equals

(a) 2 (b) 4

(c) 0 (d) 6

2. $\lim\limits_{x\to 0} x\log_e(\sin x)$ is equal to

(a) -1 (b) $\log_e 1$

(c) 1 (d) None of these

3. If the normal to the curve $y = f(x)$ at $x = 0$ is given by the equation $3x - y + 3 = 0$, then the value of $\lim\limits_{x\to 0} x^2\{f(x^2) - 5f(4x^2) + 4f(7x^2)\}^{-1}$ is

(a) $-\dfrac{1}{3}$ (b) $\dfrac{1}{3}$

(c) $-\dfrac{1}{5}$ (d) $\dfrac{1}{4}$

1. (b) **2.** (b) **3.** (a)

2 $\lim\limits_{x\to 0}\dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ is

(a) -1 (b) 1

(c) 0 (d) non-existent

(d) ***Step I*** Firstly, calculate the value of left hand limit at given value of $x = a$ for the function $f(x)$ by using the formula LHL $= \lim\limits_{x\to a^-} f(x)$.

$\therefore$ LHL $= \lim\limits_{x\to 0^-}\dfrac{e^{1/x} - 1}{e^{1/x} + 1} = \lim\limits_{h\to 0}\dfrac{e^{-1/h} - 1}{e^{-1/h} + 1}$

$= \lim\limits_{h\to 0}\left(\dfrac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1}\right)$ $\left[\because e^{1/h}\to\infty\Rightarrow\dfrac{1}{e^{1/h}}\to 0\right]$

$= -1$

Step II Now, calculate the value of right hand limit by using the formula RHL $= \lim\limits_{x\to a^+} f(x)$.

$\therefore$ RHL $= \lim\limits_{x\to 0^+}\dfrac{e^{1/x} - 1}{e^{1/x} + 1} = \lim\limits_{h\to 0}\dfrac{e^{1/h} - 1}{e^{1/h} + 1} = \lim\limits_{h\to 0}\left(\dfrac{1 - 1/e^{1/h}}{1 + 1/e^{1/h}}\right)$

$= \dfrac{1 - 0}{1 + 0} = 1$

Step III If LHL = RHL, then limit exists and its value is equal to the value of LHL or RHL.

If LHL $\neq$ RHL, then limit does not exist.

Clearly, LHL $\neq$ RHL

Hence, limit does not exist.

1. The value of $\lim\limits_{x\to k^-} x - [x]$, where k is an integer, is

(a) -1 (b) 1 (c) 0 (d) 2

2. If $f(x) = x\dfrac{e^{[x] + |x|} - 2}{[x] + |x|}$, then $\lim\limits_{x\to 0} f(x)$ is

(a) -1 (b) 0 (c) 1 (d) non-existent

3. If $f(x) = \begin{cases}\dfrac{\sin\{\cos x\}}{x - \pi/2}, & x\neq\pi/2\\ 1, & x = \pi/2\end{cases}$, where $\{\cdot\}$ represents the fractional part function, then $\lim\limits_{x\to\pi/2} f(x)$ is

(a) -1 (b) 1 (c) non-existent (d) None of these

1. (b) **2.** (d) **3.** (c)

3 Let $f(x)=\begin{cases}\dfrac{\tan x-\cot x}{x-\pi/4}, & x\neq \pi/4\\ a, & x=\pi/4\end{cases}$

Then, the value of a so that $f(x)$ is continuous at $x=\pi/4$, is

(a) 2 (b) 4

(c) 3 (d) 1

(b) ***Step I*** Find the value of $\lim\limits_{x\to a} f(x)$.

$$\lim_{x\to \pi/4} f(x)=\lim_{x\to \pi/4}\frac{\tan x-\cot x}{x-\frac{\pi}{4}}$$

$$=\lim_{x\to \pi/4}\frac{\sin^2 x-\cos^2 x}{\left(x-\frac{\pi}{4}\right)\sin x\cos x}$$

$$=\lim_{x\to \pi/4}\frac{-\cos 2x}{\left(x-\frac{\pi}{4}\right)\sin x\cos x}$$

$$=2\lim_{x\to \pi/4}\frac{\sin 2\left(\frac{\pi}{4}-x\right)}{2\left(\frac{\pi}{4}-x\right)}\times\frac{1}{\sin x\cos x}$$

$$=2\times 1\times 2=4$$

Step II Put $f(c)=\lim\limits_{x\to c} f(x)$

$\therefore$ $a=4$

1. If $f(x)=\begin{cases}\dfrac{\sqrt{1+px}-\sqrt{1-px}}{x}, & -1\le x<0\\ \dfrac{2x+1}{x-2}, & 0\le x\le 1\end{cases}$

is continuous in the interval $[-1, 1]$, then p is equal to

(a) -1 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

2. If the function $f(x)=\begin{cases}\dfrac{x^2}{a}, & 0\le x<1\\ a, & 1\le x<\sqrt{2}\\ \dfrac{2b^2-4b}{x^2}, & \sqrt{2}\le x<\infty\end{cases}$

is continuous for $0\le x<\infty$, then the most suitable values of a and b are

(a) $a=1, b=-1$ (b) $a=-1, b=1+\sqrt{2}$

(c) $a=-1, b=1$ (d) None of these

3. If $f(x)=\begin{cases}\dfrac{1-\sqrt{2}\sin x}{\pi-4x}, & x\neq\dfrac{\pi}{4}\\ a, & x=\dfrac{\pi}{4}\end{cases}$ is continuous at $x=\dfrac{\pi}{4}$,

then a equals

(a) 4 (b) 2 (c) 1 (d) 1/4

1. (b) **2.** (c) **3.** (d)

4 $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ equals

(a) e (b) ∞

(c) e^2 (d) e^{-1}

(c) Step I Using direct substitution, find the value of limit.

Given limit $=\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ $[1^\infty$ form$]$

Step II If value of limit is 1^∞, then write the function in the form of $(1+f(x))^{g(x)}$, if it is not $\lim\limits_{x\to\infty}\left(1+\frac{2}{x}\right)^x$.

Step III Use formula

$$\lim_{x\to a}(1+f(x))^{g(x)}=e^{\lim\limits_{x\to a} f(x)\cdot g(x)}$$

$\therefore$ $$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=e^{\lim\limits_{x\to\infty}\frac{2}{x}\cdot x}=e^2$$

1. $\lim\limits_{x\to 0}(\cos x)^{\cot x}$ equals

(a) e (b) e^{-1}

(c) 1 (d) -1

2. The value of $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$, is

(a) $e^{-1/\pi}$ (b) $e^{2/\pi}$

(c) $e^{-2/\pi}$ (d) $e^{1/\pi}$

3. The value of $\lim\limits_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$, is

(a) e (b) e^2

(c) $2e^{-1}$ (d) e^{-2}

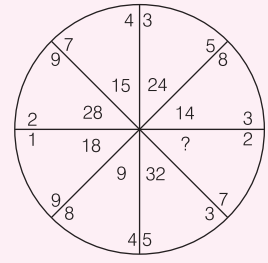
4. $\lim\limits_{x\to\infty}\left(1-\frac{4}{x-1}\right)^{3x-1}$ is equal to

(a) e^{12} (b) e^{-12}

(c) e^4 (d) e^3

1. (c) **2.** (c) **3.** (b) **4.** (b)

GUESS The Number!!!



Er. Love Agrawal

JEE ADVANCED DRILL

A collection of 30 innovative problems on JEE Advanced pattern from complete syllabus

Single Option Correct Type

1. The value of $\cot\left\{\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}$ is

(a) $\frac{23}{25}$ *(b)* $\frac{25}{23}$

(c) $\frac{23}{24}$ *(d)* $\frac{24}{23}$

2. The centroid of an equilateral triangle is (0, 0). If two vertices of the triangle lies on $x+y=2\sqrt{2}$, then one of them will have its coordinates

(a) $(\sqrt{2}+\sqrt{6},\sqrt{2}-\sqrt{6})$

(b) $(\sqrt{2}+\sqrt{3},\sqrt{2}-\sqrt{3})$

(c) $(\sqrt{2}+\sqrt{5},\sqrt{2}-\sqrt{5})$

(d) None of the above

3. The scalar of $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})\times(\mathbf{A}+\mathbf{B}+\mathbf{C})$ equals

(a) 0 *(b)* $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]+[\mathbf{B}\ \mathbf{C}\ \mathbf{A}]$

(c) $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]$ *(d)* None of these

4. $\lim_{x\to\infty}\frac{x(\log x)^3}{1+x+x^2}$ equals

(a) 0 *(b)* −1

(c) 1 *(d)* None of these

5. Let $f'(x)=\frac{192\,x^3}{2+\sin^4\pi x}$ for all $x\in R$ with $f\left(\frac{1}{2}\right)=0$. If $m\le\int_{1/2}^{1}f(x)\,dx\le M$, then

(a) $m=13,\ M=24$

(b) $m=\frac{1}{4},\ M=\frac{1}{2}$

(c) $m=-11,\ M=0$

(d) $m=\frac{26}{10},\ M=\frac{78}{20}$

One or More than One Option Correct Type

6. If $z_1=5+12i$ and $|z_2|=4$, then

(a) $\max(|z_1+iz_2|)=17$ *(b)* $\min(|z_1+(1+i)z_2|)=13-4\sqrt{2}$

(c) $\min\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{4}$ *(d)* $\max\left|\frac{z_1}{z_2+\frac{4}{z_2}}\right|=\frac{13}{3}$

7. If $f(x)=\begin{vmatrix}3 & 3x & 3x^2+2a^2\\ 3x & 3x^2+2a^2 & 3x^3+6a^2x\\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^2\end{vmatrix}$, then

(a) $f'(x)=0$

(b) $y=f(x)$ is a straight line parallel to X-axis

(c) $\int_0^2 f(x)\,dx=32a^4$

(d) None of the above

8. Let $(1+x^2)^2(1+x)^n=\sum_{k=0}^{n+4}a_kx^k$. If a_1, a_2 and a_3 are in AP, then the possible value(s) of n is/are

(a) 5 *(b)* 4 *(c)* 3 *(d)* 2

9. If x, y and z are positive numbers in AP, then

(a) $y^2\ge xz$ *(b)* $xy+yz\ge 2xz$

(c) $\frac{x+y}{2y-x}+\frac{y+z}{2y-z}\ge 4$ *(d)* None of these

10. If a circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2+y^2=16$ and $x^2+y^2=1$, then

(a) radius of S is 8 *(b)* radius of S is 7

(c) centre of S is $(-7,1)$ *(d)* centre of S is $(-8,1)$

11. Suppose $f(x)$ and $g(x)$ are functions having second derivatives $f''(x)$ and $g''(x)$ everywhere. If $f(x)\cdot g(x)=1$ for all x and $f'(x), g'(x)$ are never zero, then $\dfrac{f''(x)}{f'(x)}-\dfrac{g''(x)}{g'(x)}$ equals

(a) $\dfrac{-2f'(x)}{f(x)}$ (b) $-\dfrac{2g'(x)}{g(x)}$ (c) $-\dfrac{f'(x)}{f(x)}$ (d) $\dfrac{2f'(x)}{f(x)}$

12. Let $P(k)=\left(1+\cos\dfrac{\pi}{4k}\right)\left(1+\dfrac{\cos(2k-1)\pi}{4k}\right)\left(1+\dfrac{\cos(2k+1)\pi}{4k}\right)\left(\dfrac{1+\cos(4k-1)\pi}{4k}\right)$, then

(a) $P(3)=\dfrac{1}{16}$ (b) $P(4)=\dfrac{2-\sqrt{2}}{16}$

(c) $P(5)=\dfrac{3-\sqrt{5}}{32}$ (d) $P(6)=\dfrac{2-\sqrt{3}}{16}$

13. If a rod of length 2 units whose one end is $(1, 0, -1)$ and other end touches the plane $x-2y+2z+4=0$, then

(a) the rod sweeps the figure whose volume is π cu units

(b) the area of the region in which the rod traces on the plane, is 2π sq units

(c) the length of projection of the rod on the plane is $\sqrt{3}$ units

(d) the centre of the region in which the rod traces on the plane, is $\left(\dfrac{2}{3},\dfrac{2}{3},-\dfrac{5}{3}\right)$

Single Integer Type Questions

14. Let α and β be the solutions of the quadratic equation $x^2-1154x+1=0$, then the value of $\sqrt[4]{\alpha}+\sqrt[4]{\beta}$ is ..... .

15. For any $x, y\in R$ and $xy>0$, the minimum value of $\dfrac{2x}{y^3}+\dfrac{x^3y}{3}+\dfrac{4y^2}{9x^4}$ is ..... .

16. If A is a square matrix of order 3 such that $|A|=2$, then $|(\text{adj}\,A^{-1})^{-1}|$ equals ..... .

17. Suppose that all the terms of an AP are natural numbers. If the ratio of the sum of first seven terms to the sum of first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this AP is ..... .

18. A tangent drawn to hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at $P\left(\dfrac{\pi}{6}\right)$ forms a triangle of area $3a^2$ sq units, with the coordinate axes. If the eccentricity of the hyperbola is e, then the value of e^2-9 is ..... .

19. If the sum of slopes of the lines given by $x^2-2cxy-7y^2=0$ is four times their product, then the value of c is ..... .

20. In ΔABC, if $\angle C=2\angle A$ and $AC=2BC$, then the value of $\dfrac{a^2+b^2+c^2}{R^2}$, where R is circumradius of triangle, is ..... .

21. If the solution of the differential equation $\dfrac{dy}{dx}=\dfrac{1}{x\cos y+\sin 2y}$ is $x=Ce^{\sin y}-k(1+\sin y)$, then the value of k is ..... .

Linked Comprehension Type Questions

Passage (Q. Nos. 22-23) Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls while U_2 contains only 1 white ball. A fair coin is tossed. If head appears, then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2. Now, 1 ball is drawn at random from U_2.

22. The probability of the drawn ball from U_2 being white is

(a) $\dfrac{13}{30}$ (b) $\dfrac{23}{30}$ (c) $\dfrac{19}{30}$ (d) $\dfrac{11}{30}$

23. If drawn ball from U_2 is white, then the probability that head appeared on the coin is

(a) $\dfrac{17}{23}$ (b) $\dfrac{11}{23}$ (c) $\dfrac{15}{23}$ (d) $\dfrac{12}{23}$

Passage (Q. Nos. 24-25) Consider one side AB of a square $ABCD$ in order on line $y=2x-17$ and other two vertices C, D on $y=x^2$.

24. The minimum intercept of line CD on the Y-axis is

(a) 3 (b) 4 (c) 2 (d) 6

25. The maximum possible area of square $ABCD$ is

(a) 1180 (b) 1250

(c) 1280 (d) None of these

Passage (Q. Nos. 26-27) Let $f'(\sin x)<0$ and $f''(\sin x)>0, \forall\, x\in\left(0,\dfrac{\pi}{2}\right)$ and $g(x)=f(\sin x)+f(\cos x)$.

26. Which one of the following is true?

(a) $g'(x)$ is increasing

(b) $g'(x)$ is decreasing

(c) $g'(x)$ has a point of minima

(d) $g'(x)$ has a point of maxima

27. Which one of the following is true?

(a) $g(x)$ is decreasing in $\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

(b) $g(x)$ is increasing in $\left(0,\dfrac{\pi}{4}\right)$

(c) $g(x)$ is monotonically increasing

(d) None of the above

Match the Columns

28. Match the column.

Column I	Column II
A. Four dice (six faced) are rolled. The number of possible outcomes in which atleast one die shows 2, is	p. 210
B. Let A be the set of 4-digit numbers $a_1a_2a_3a_4$, where $a_1 > a_2 > a_3 > a_4$. Then, $n(A)$ is equal to	q. 480
C. The total number of 3-digit numbers, the sum of whose digits is even, is equal to	r. 671
D. The number of 4-digit numbers that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7 so that each number contains digit 1, is	s. 450

29. Match the column.

Column I (Equation)	Column II (Solution)
A. $\cos^2 2x + \cos^2 x = 1$	p. $x = \left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{\pi}{6}\right\}$, $n \in Z$
B. $\cos x + \sqrt{3}\sin x = \sqrt{3}$	q. $x = \frac{n\pi}{3}$, $n \in Z$
C. $1 + \sqrt{3}\tan^2 x = (1 + \sqrt{3})\tan x$	r. $x = (2n - 1)\frac{\pi}{6}$, $n \in Z$
D. $\tan 3x - \tan 2x - \tan x = 0$	s. $x = \left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\}$, $n \in Z$

30. Match the column.

Column I	Column II
A. The coordinates of a point on the lines $x = 4y + 5, z = 3y - 6$ at a distance 3 units from the point $(5, 3, -6)$, is/are	p. $(-1, -2, 0)$
B. The plane containing the line $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$ has the point	q. $(5, 0, -6)$
C. A line passes through two points $A(2, -3, -1)$ and $B(8, -1, 2)$. The coordinates of a point on this line nearer to the origin and at a distance of 14 units from A, is/are	r. $(2, 5, 7)$
D. The coordinates of the foot of the perpendicular from the point $(3, -1, 11)$ on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$, is/are	s. $(14, 1, 5)$

Answers with Explanation

1. ***(b)*** We have, $\cot\left\{\sum_{n=1}^{23}\cot^{-1}\left(1 + \sum_{k=1}^{n}2k\right)\right\}$

$= \cot\left\{\sum_{n=1}^{23}\cot^{-1}[1 + n(n+1)]\right\}$

$= \cot\left\{\sum_{n=1}^{23}\cot^{-1}(n^2 + n + 1)\right\}$

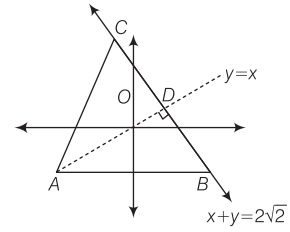
$= \cot\left\{\sum_{n=1}^{23}\tan^{-1}\frac{(n+1) - n}{1 + n(n+1)}\right\}$

$= \cot\left\{\sum_{n=1}^{23}(\tan^{-1}(n+1) - \tan^{-1}n)\right\}$

$= \cot(\tan^{-1}24 - \tan^{-1}1) = \cot\left(\tan^{-1}\frac{23}{25}\right) = \frac{25}{23}$

2. ***(a)*** Let the vertices B and C lie on the given line.

Then, $OD = \frac{2\sqrt{2}}{\sqrt{2}} = 2$



The equation of OD is $y = x$ or $x = y = \sqrt{2}$ (for point D).

Also, $BD = OD\tan 60° = 2\sqrt{3}$

By using parametric equation of line, we get

$$\frac{x - \sqrt{2}}{1/\sqrt{2}} = \frac{y - \sqrt{2}}{1/\sqrt{2}} = \pm 2\sqrt{3}$$

$\therefore$ $C \equiv (\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6})$

and $B \equiv (\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6})$

3. ***(a)*** We have, $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) \times (\mathbf{A} + \mathbf{B} + \mathbf{C})$

$= \mathbf{A}\cdot[\mathbf{B}\times\mathbf{A} + \mathbf{B}\times\mathbf{B} + \mathbf{B}\times\mathbf{C} + \mathbf{C}\times\mathbf{A} + \mathbf{C}\times\mathbf{B} + \mathbf{C}\times\mathbf{C}]$

$= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{A}) + \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) + \mathbf{A}\cdot(\mathbf{C}\times\mathbf{A}) + \mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$

$= [\mathbf{A}\ \mathbf{B}\ \mathbf{C}] + [\mathbf{A}\ \mathbf{C}\ \mathbf{B}]$

$= [\mathbf{A}\ \mathbf{B}\ \mathbf{C}] - [\mathbf{A}\ \mathbf{B}\ \mathbf{C}] = 0$

4. ***(a)*** We have, $\lim_{x\to\infty}\frac{x(\log x)^3}{1 + x + x^2}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$= \lim_{x\to\infty}\frac{3(\log x)^2 + (\log x)^3}{1 + 2x}$ [using L' Hospital's rule]

$= \lim_{x\to\infty}\frac{6(\log x)\cdot\frac{1}{x} + 3(\log x)^2\cdot\frac{1}{x}}{2}$

$= \lim_{x\to\infty}\frac{6\log x + 3(\log x)^2}{2x} = \lim_{x\to\infty}\frac{\frac{6}{x} + 3\times 2\log x\cdot\frac{1}{x}}{2}$

$= \lim_{x\to\infty}\frac{6 + 6\log x}{2x} = \lim_{x\to\infty}\frac{\frac{6}{x}}{2} = \lim_{x\to\infty}\frac{3}{x} = 3\times 0 = 0$

5. ***(d)*** We have, $f'(x) = \dfrac{192x^3}{2 + \sin^4 \pi x}, \forall\ x \in R; f\left(\dfrac{1}{2}\right) = 0$

Now, $\quad 64x^3 \le f'(x) \le 96x^3, \forall\ x \in \left[\dfrac{1}{2}, 1\right]$

$\Rightarrow \int_{1/2}^{x} 64x^3 dx \le \int_{1/2}^{x} f'(x)dx \le \int_{1/2}^{x} 96x^3 dx$

So, $\quad 16x^4 - 1 \le f(x) \le 24x^4 - \dfrac{3}{2}, \forall\ x \in \left[\dfrac{1}{2}, 1\right]$

$\Rightarrow \int_{1/2}^{1} (16x^4 - 1)dx \le \int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1} \left(24x^4 - \dfrac{3}{2}\right)dx$

$\Rightarrow \quad \dfrac{16}{5} \cdot \dfrac{31}{32} - \dfrac{1}{2} \le \int_{1/2}^{1} f(x)dx \le \dfrac{24}{5} \cdot \dfrac{31}{32} - \dfrac{3}{4}$

$\Rightarrow \quad \dfrac{26}{10} \le \int_{1/2}^{1} f(x)dx \le \dfrac{78}{20}$

6. ***(a, b, d)*** We have, $z_1 = 5 + 12i, |z_2| = 4$

$\therefore \quad |z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\therefore \quad |z_1 + (1 + i)z_2| \ge ||z_1| - |1 + i||z_2|| = 13 - 4\sqrt{2}$

$\therefore \quad \min(|z_1 + (1 + i)z_2|) = 13 - 4\sqrt{2}$

Again, $\quad \left|z_2 + \dfrac{4}{z_2}\right| \le |z_2| + \dfrac{4}{|z_2|} = 4 + 1 = 5$

and $\quad \left|z_2 + \dfrac{4}{z_2}\right| \ge |z_2| - \dfrac{4}{|z_2|} = 4 - 1 = 3$

$\therefore \quad \max\left|\dfrac{z_1}{z_2 + \dfrac{4}{z_2}}\right| = \dfrac{13}{3}$ and $\min\left|\dfrac{z_1}{z_2 + \dfrac{4}{z_2}}\right| = \dfrac{13}{5}$

7. ***(a, b)*** We have,

$$f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2x & 3x^4 + 12a^2x^2 + 2a^2 \end{vmatrix}$$

Applying $C_3 \to C_3 - xC_2, C_2 \to C_2 - xC_1$, we get

$$f(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2 + 2a^2 & 4a^2x & 6a^2x^2 + 2a^2 \end{vmatrix}$$

Applying $C_3 \to C_3 - xC_2$, we get

$$f(x) = 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2 + 2a^2 & 2x & x^2 + 2a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3C_3$, we get

$$f(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -4a^2 & 2x & x^2 + 2a^2 \end{vmatrix} = 16a^6$$

8. ***(b, c, d)*** LHS

$= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \ldots)$

RHS $= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

On comparing the coefficients of $x, x^2, x^3, \ldots$, we get

$a_1 = C_1, a_2 = C_2 + 2, a_3 = C_3 + 2C_1 \quad \ldots$(i)

Now, $2a_2 = a_1 + a_3$ which forms an AP.

$\Rightarrow \quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$ [using Eq. (i)]

$\Rightarrow \quad \dfrac{2n(n-1)}{2} + 4 = 3n + \dfrac{n(n-1)(n-2)}{6}$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n - 2)(n^2 - 7n + 12) = 0$

$\Rightarrow \quad (n - 2)(n - 3)(n - 4) = 0$

$\therefore \quad n = 2, 3, 4$

9. ***(a, b, c)*** AM of x, z is y and GM of x, z is $\sqrt{xz}$.

Now, $\quad$ AM $\ge$ GM

$\Rightarrow \quad y \ge \sqrt{xz} \Rightarrow y^2 \ge xz$

Also, $\quad$ AM $\ge$ HM

$\Rightarrow \quad y \ge \dfrac{2xz}{x + z}$

$\Rightarrow \quad xy + yz \ge 2xz$

Again, $\quad \dfrac{x + y}{2y - x} = \dfrac{x + y}{x + z - x} = \dfrac{x + y}{z}$

and $\quad \dfrac{y + z}{2y - z} = \dfrac{y + z}{x + z - z} = \dfrac{y + z}{x}$

$\therefore \quad \dfrac{\dfrac{x + y}{2y - x} + \dfrac{y + z}{2y - z}}{2} \ge \sqrt{\dfrac{x + y}{z} \cdot \dfrac{y + z}{x}}$

$= \sqrt{1 + \dfrac{y(x + y + z)}{xz}} = \sqrt{1 + \dfrac{3y^2}{xz}}$

$\therefore \quad \dfrac{x + y}{2y - x} + \dfrac{y + z}{2y - z} \ge 2\sqrt{1 + \dfrac{3y^2}{xz}} \ge 4 \qquad [\because y^2 \ge xz]$

10. ***(b, c)*** Given circles are

$S_1 : x^2 + y^2 - 2x - 15 = 0$

and $\quad S_2 : x^2 + y^2 - 1 = 0$

Centre of the circle which intersects above two circles orthogonally lies on the radial axis of the circles which is

$S_1 - S_2 = 0$ or $x + 7 = 0$.

Let the centre of the required circle be $C(-7, k)$.

Circle passes through the point $A(0, 1)$.

$\therefore$ Radius, $r = \sqrt{7^2 + (k - 1)^2}$

Also, radius = length of the tangent from C to anyone of the given circles.

$\therefore \quad r = \sqrt{7^2 + k^2 - 1}$

$\therefore \quad 7^2 + (k - 1)^2 = 7^2 + k^2 - 1$

$\Rightarrow \quad -2k + 1 = -1 \Rightarrow k = 1$

$\therefore \quad r = 7$

11. ***(b, d)*** We have, $g(x) = \dfrac{1}{f(x)}$

$\therefore \quad g'(x) = \dfrac{-1}{f^2(x)} f'(x)$

or $\quad g''(x) = -\left[\dfrac{-2}{f^3(x)} f'^2(x) + \dfrac{1}{f^2(x)} f''(x)\right]$

$= \dfrac{2}{f^3(x)} f'^2(x) - \dfrac{f''(x)}{f^2(x)}$

$\Rightarrow \quad \dfrac{f''(x)}{f'(x)} - \dfrac{g''(x)}{g'(x)} = \dfrac{f''(x)}{f'(x)} - \dfrac{\dfrac{2}{f^3(x)} f'^2(x) - \dfrac{f''(x)}{f^2(x)}}{\dfrac{-1}{f^2(x)} f'(x)}$

$$= \frac{f''(x)}{f'(x)} - \left(\frac{-2f'(x)}{f(x)} + \frac{f''(x)}{f'(x)}\right) = \frac{2f'(x)}{f(x)}$$

Also, $g(x) \cdot f(x) = 1$

or $g'(x) f(x) + g(x) f'(x) = 0$

$\Rightarrow \quad \frac{f'(x)}{f(x)} = -\frac{g'(x)}{g(x)} = 0$

$\therefore \quad \frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)} = \frac{2f'(x)}{f(x)} = \frac{-2g'(x)}{g(x)}$

12. ***(a, b, c, d)*** $P(k) = \left(1 + \cos\frac{\pi}{4k}\right)\left[1 + \cos\left(\frac{\pi}{2} - \frac{\pi}{4k}\right)\right]$

$$\left[1 + \cos\left(\frac{\pi}{2} + \frac{\pi}{4k}\right)\right]\left[1 + \cos\left(\pi - \frac{\pi}{4k}\right)\right]$$

$$= \left(1 + \cos\frac{\pi}{4k}\right)\left(1 + \sin\frac{\pi}{4k}\right)\left(1 - \sin\frac{\pi}{4k}\right)\left(1 - \cos\frac{\pi}{4k}\right)$$

$$= \left(1 - \cos^2\frac{\pi}{4k}\right)\left(1 - \sin^2\frac{\pi}{4k}\right)$$

$$= \sin^2\frac{\pi}{4k}\cos^2\frac{\pi}{4k} = \frac{4\sin^2\frac{\pi}{4k}\cos^2\frac{\pi}{4k}}{4} = \frac{1}{4}\sin^2\frac{\pi}{2k}$$

$\therefore \quad P(3) = \frac{1}{4}\sin^2\frac{\pi}{6} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

$$P(4) = \frac{1}{4}\sin^2\frac{\pi}{8} = \frac{1}{8}\left(1 - \cos\frac{\pi}{4}\right) = \frac{2 - \sqrt{2}}{16}$$

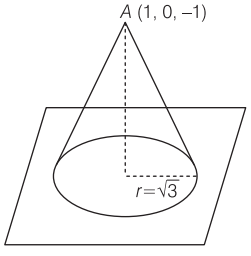
$$P(5) = \frac{1}{4}\sin^2\frac{\pi}{10} = \frac{1}{8}\left(2\sin^2\frac{\pi}{10}\right) = \frac{1}{8}(1 - \cos 36^\circ)$$

$$= \frac{1}{8}\left(1 - \frac{\sqrt{5} + 1}{4}\right) = \frac{3 - \sqrt{5}}{32}$$

$$P(6) = \frac{1}{4}\sin^2\frac{\pi}{12} = \frac{1}{8}\left(2\sin^2\frac{\pi}{12}\right)$$

$$= \frac{1}{8}\left(1 - \cos\frac{\pi}{6}\right) = \frac{1}{8}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2 - \sqrt{3}}{16}$$

13. ***(a, c, d)*** The rod sweeps out the figure which is a cone.



The distance of point $A(1, 0, -1)$ from the plane

$$= \frac{|1 - 2 + 4|}{\sqrt{9}} = 1 \text{ unit}$$

The slant height l of the cone is 2 units.

Then, the radius of the base of the cone is

$$\sqrt{l^2 - 1} = \sqrt{4 - 1} = \sqrt{3}$$

Hence, the volume of the cone

$$= \frac{\pi}{3}(\sqrt{3})^2 \cdot 1 = \pi \text{ cu units}$$

Area of the circle on the plane in which the rod traces, is 3π sq units.

Also, the centre of the circle is $Q(x, y, z)$.

Then, $\frac{x - 1}{1} = \frac{y - 0}{-2} = \frac{z + 1}{2} = \frac{-(1 - 0 - 2 + 4)}{1^2 + (-2)^2 + 2^2}$

$\Rightarrow \quad Q(x, y, z) \equiv \left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

14. ***(6)*** Since, α and β are the roots of the equation

$$x^2 - 1154x + 1 = 0.$$

$\therefore \quad \alpha + \beta = 1154$ and $\alpha\beta = 1$

$\Rightarrow \quad (\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta}$

$$= 1154 + 2 = 1156$$

$\Rightarrow \quad \sqrt{\alpha} + \sqrt{\beta} = 34$

Again, $(\alpha^{1/4} + \beta^{1/4})^2 = \sqrt{\alpha} + \sqrt{\beta} + 2(\alpha\beta)^{1/4}$

$$= 34 + 2 = 36$$

$\Rightarrow \quad \alpha^{1/4} + \beta^{1/4} = 6$

15. ***(2)*** As $x, y \in R$ and $xy > 0$, so x and y will be of same sign.

Therefore, all the quantities $\frac{2x}{y^3}, \frac{x^3y}{3}$ and $\frac{4y^2}{9x^4}$ are positive.

Now, AM $\geq$ GM

$\therefore \quad \frac{2x}{y^3} + \frac{x^3y}{3} + \frac{4y^2}{9x^4} \geq 3\left(\frac{2x}{y^3} \cdot \frac{x^3y}{3} \cdot \frac{4y^2}{9x^4}\right)^{1/3} \geq 3 \times \frac{2}{3} \geq 2$

16. ***(4)*** $|\text{adj } A^{-1}| = |A^{-1}|^2 = \frac{1}{|A|^2}$

$\Rightarrow \quad |(\text{adj } A^{-1})^{-1}| = \frac{1}{|\text{adj } A^{-1}|} = |A|^2 = (2)^2 = 4$

17. ***(9)*** Given, $\frac{S_7}{S_{11}} = \frac{6}{11}$

$\Rightarrow \quad \frac{\frac{7}{2}[2a + 6d]}{\frac{11}{2}[2a + 10d]} = \frac{6}{11}$

$\Rightarrow \quad \frac{7(a + 3d)}{11(a + 5d)} = \frac{6}{11}$

$\Rightarrow \quad 7a + 21d = 6a + 30d$

$\Rightarrow \quad a = 9d$

$\Rightarrow \quad T_7 = a + 6d = 15d$

Given, $130 < T_7 < 140 \Rightarrow 130 < 15d < 140$

$\Rightarrow \quad d = 9$

18. ***(8)*** The point $P(\pi/6)$ is $\left(a\sec\frac{\pi}{6}, b\tan\frac{\pi}{6}\right)$

$\therefore \quad P \equiv \left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

Therefore, the equation of tangent at P is

$$\frac{x}{\sqrt{3}\,a/2} - \frac{y}{\sqrt{3}b} = 1$$

$\therefore$ Area of triangle $= \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$

$\therefore \quad \frac{b}{a} = 4$

$\therefore \quad e^2 = 1 + \frac{b^2}{a^2} = 17$

$\Rightarrow \quad e^2 - 9 = 17 - 9 = 8$

19. *(2)* We have,

$$m_1 + m_2 = 4m_1m_2$$

$$\Rightarrow \quad \frac{-2h}{b} = \frac{4a}{b} \Rightarrow \frac{-2(-c)}{-7} = \frac{4\times 1}{-7}$$

$$\Rightarrow \quad 2c = 4 \Rightarrow c = 2$$

20. *(8)* We know that,

$$A + B + C = \pi$$

$$\Rightarrow \quad A + B + 2A = \pi \qquad [\because C = 2A]$$

$$\Rightarrow \quad B = \pi - 3A$$

As, $0 < C < \pi \Rightarrow 0 < 2A < \pi$

$$\Rightarrow \quad 0 < A < \frac{\pi}{2}$$

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{2a}{\sin(\pi - 3A)} \qquad [\because AC = 2BC]$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{2a}{\sin 3A}$$

$$\Rightarrow \quad \frac{1}{\sin A} = \frac{2}{3\sin A - 4\sin^3 A}$$

$$\Rightarrow \quad 3 - 4\sin^2 A = 2 \Rightarrow \sin^2 A = \frac{1}{4}$$

$$\Rightarrow \quad \sin A = \frac{1}{2} \Rightarrow A = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

But $0 < A < \frac{\pi}{2}$

$$\therefore \quad A = \frac{\pi}{6}$$

Thus, $B = \frac{\pi}{2}$ and $C = \frac{\pi}{3}$

Now, $a^2 + b^2 + c^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$

$$\Rightarrow a^2 + b^2 + c^2 = 4R^2\left(\sin^2\frac{\pi}{6} + \sin^2\frac{\pi}{2} + \sin^2\frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{R^2} = 4\left(\frac{1}{4} + 1 + \frac{3}{4}\right) \Rightarrow \frac{a^2 + b^2 + c^2}{R^2} = 8$$

21. *(2)* We have, $\frac{dy}{dx} = \frac{1}{x\cos y + 2\sin y\cos y}$

$$\Rightarrow \quad \frac{dx}{dy} = x\cos y + 2\sin y\cos y$$

$$\Rightarrow \quad \frac{dx}{dy} + (-\cos y)x = 2\sin y\cos y$$

$$\therefore \text{ If } = e^{\int -\cos y\, dy} = e^{-\sin y}$$

Thus, the solution is

$$xe^{-\sin y} = 2\int e^{-\sin y}\sin y\cos y\, dy$$

$$\Rightarrow \quad xe^{-\sin y} = -2\sin y e^{-\sin y} - 2\int(-e^{-\sin y})\cos y\, dy$$

$$\Rightarrow \quad xe^{-\sin y} = -2\sin y e^{-\sin y} + 2\int e^{-\sin y}\cos y\, dy$$

$$\Rightarrow \quad xe^{-\sin y} = -2\sin y e^{-\sin y} - 2e^{-\sin y} + C$$

$$\Rightarrow \quad x = -2\sin y - 2 + Ce^{\sin y}$$

$$\Rightarrow \quad x = Ce^{\sin y} - 2(1 + \sin y)$$

$$\therefore \quad k = 2$$

22. *(b)* Let H : 1 ball drawn from U_1 to U_2

T : 2 balls drawn from U_1 to U_2 and E : 1 ball drawn from U_2

$$\therefore \ P(W \text{ from } U_2) = \frac{1}{2}\times\left(\frac{3}{5}\times 1\right) + \frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right)$$

$$+ \frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{{}^3C_1\cdot{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right) = \frac{23}{30}$$

23. *(d)* $P\left(\frac{H}{W}\right) = \frac{P(W/H)\times P(H)}{P(W/T)\cdot P(T) + P(W/H)\cdot P(H)}$

$$= \frac{\frac{1}{2}\left(\frac{3}{2}\times 1 + \frac{2}{5}\times\frac{1}{2}\right)}{23/30} = \frac{12}{23}$$

Sol. (Q. Nos. 24-25) Let the equation of line CD be

$$y = 2x + b \qquad \ldots(i)$$

and the length of side of square be a.

Let the coordinates of C and D be (x_1, y_1) and (x_2, y_2), respectively. Therefore, $\frac{y_2 - y_1}{x_2 - x_1} = 2 \qquad [\because CD \| AB]$

Also, $a^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2 = 5(x_2 - x_1)^2$

or $a^2 = 5[(x_1 + x_2)^2 - 4x_1x_2] \qquad \ldots(ii)$

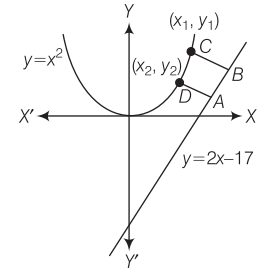
On solving Eq. (i) with $y = x^2$, we get

$$x^2 = 2x + b \text{ or } x_1 + x_2 = 2,\ x_1x_2 = -b$$

On putting these values in Eq. (ii), we get

$$a^2 = 20(b + 1) \qquad \ldots(iii)$$

Now, distance CD is



$$a = \left|\frac{2x_1 - y_1 - 17}{\sqrt{5}}\right| = \left|\frac{-b - 17}{\sqrt{5}}\right| \qquad \text{[using Eq. (i)]}$$

or $5a^2 = (b + 17)^2 \qquad \ldots(iv)$

From Eqs. (iii) and (iv), we get

$$100(b + 1) = (b + 17)^2 \Rightarrow b = 3 \text{ or } b = 63$$

$$\therefore \quad a^2 = 80 \text{ or } a^2 = 1280$$

24. *(a)* **25.** *(c)*

Sol. (Q. Nos. 26-27) Here, $g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

or $g''(x) = -f'(\sin x)\sin x + \cos^2 x\, f''(\sin x)$

$$+ f''(\cos x)\sin^2 x - f'(\cos x)\cos x > 0, \forall\ x \in (0, \pi/2)$$

$[\because f'(\sin x) = f'(\cos x(\pi/2 - x)) < 0$ and

$f''(\sin x) = f''(\cos x(\pi/2 - x)) > 0]$

Thus, $g'(x)$ is increasing in $(0, \pi/2)$. Also, $g'(\pi/4) = 0$

or $g'(x) > 0, \forall\ x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0, \forall\ x \in (0, \pi/4)$

Thus, $g(x)$ is decreasing in $(0, \pi/4)$.

26. *(a)* **27.** *(d)*

28. A → r; B → p; C → s; D → q

A. The number of possible outcomes with 2 on atleast one dice
= (Total number of outcomes) – (Number of outcomes in which 2 does not appear on any dice)
$= 6^4 - 5^4 = 1296 - 625 = 671$

B. Any selection of four digits from 10 digits 0, 1, 2, 3, …, 9 gives one number.
So, the required number of numbers $= {}^{10}C_4$
$$= \frac{10!}{(10-4)!\,4!} = \frac{10\times 9\times 8\times 7\times 6!}{6!\times 4\times 3\times 2}$$
$$= 210$$

C. Let the number be $n = pqr$. Since, $p+q+r$ is even, so p can be filled in 9 ways and q can be filled in 10 ways.
r can be filled in number of ways depending upon what is the sum of p and q.
If $p+q$ is odd, then r can be filled with anyone of five odd digits.
If $p+q$ is even, then r can be filled with anyone of five even digits.
In any case, r can be filled in five ways.
Hence, total number of numbers $= 9\times 10\times 5 = 450$

D. After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is ${}^6P_2$.
Therefore, total number of ways $= {}^7P_3 - {}^6P_2 = 480$.

29. A → r; B → s; C → p; D → q

A. $\cos^2 2x - \sin^2 x = 0$

$\Rightarrow \cos 3x \cos x = 0 \Rightarrow \cos 3x = 0$ or $\cos x = 0$

$\Rightarrow 3x = (2n-1)\frac{\pi}{2}$ or $x = (2n-1)\frac{\pi}{2}, n \in Z$

or $x = (2n-1)\frac{\pi}{6}$ or $x = (2n-1)\frac{\pi}{2}$

Hence, the general solution is $(2n-1)\frac{\pi}{6}$ as $(2n-1)\frac{\pi}{2}$ is contained in $(2n-1)\frac{\pi}{6}$.

B. $\cos x + \sqrt{3}\sin x = \sqrt{3}$ or $\frac{\cos x}{2} + \frac{\sqrt{3}}{2}\sin x = \frac{\sqrt{3}}{2}$

or $\cos\left(x - \frac{\pi}{3}\right) = \cos\frac{\pi}{6} \Rightarrow x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}, n \in Z$

$\Rightarrow x = 2n\pi + \frac{\pi}{2}$ or $x = 2n\pi + \frac{\pi}{6}$

C. $\sqrt{3}\tan^2 x - (\sqrt{3}+1)\tan x + 1 = 0$

or $\sqrt{3}\tan x(\tan x - 1) - (\tan x - 1) = 0$

or $(\tan x - 1)(\sqrt{3}\tan x - 1) = 0$

$\therefore \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$

or $\tan x = \frac{1}{\sqrt{3}} \Rightarrow x = n\pi + \frac{\pi}{6}, n \in Z$

D. $\tan 3x - \tan 2x - \tan x = 0$

or $\tan x \tan 2x \tan 3x = 0$

$x = n\pi$ or $n\pi/2$ (rejected) or $n\pi/3$

Therefore, the general solution is $n\pi/3$ as $n\pi$ is contained in $n\pi/3$.

30. A → q; B → p; C → s; D → r

A. Given lines are $x = 4y + 5, z = 3y - 6$

or $\frac{x-5}{4} = y, \quad \frac{z+6}{3} = y$

or $\frac{x-5}{4} = \frac{y}{1} = \frac{z+6}{3} = \lambda$ [say]

Any point on the line is of the form $(4\lambda + 5, \lambda, 3\lambda - 6)$.
Since, the distance between $(4\lambda + 5, \lambda, 3\lambda - 6)$ and $(5, 3, -6)$ is 3 units.

$\therefore \quad (4\lambda + 5 - 5)^2 + (\lambda - 3)^2 + (3\lambda - 6 + 6)^2 = 9$

or $16\lambda^2 + \lambda^2 + 9 - 6\lambda + 9\lambda^2 = 9$ or $26\lambda^2 - 6\lambda = 0$

or $\lambda = 0, 3/13$

Hence, the point is $(5, 0, -6)$.

B. The equation of the plane containing the line $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ is

$$\begin{vmatrix} x-2 & y+3 & z+5 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

or $x - 2y + z - 3 = 0$

Hence, point $(-1, -2, 0)$ lies on this plane.

C. The line passing through points $A(2, -3, -1)$ and $B(8, -1, 2)$ is $\frac{x-2}{8-2} = \frac{y+3}{-1+3} = \frac{z+1}{2+1}$

or $\frac{x-2}{6} = \frac{y+3}{2} = \frac{z+1}{3} = \lambda$ [say]

Any point on this line is of the form $P(6\lambda + 2, 2\lambda - 3, 3\lambda - 1)$, whose distance from point $A(2, -3, -1)$ is 14 units.

$\therefore \quad PA = 14 \quad \text{or} \quad PA^2 = (14)^2$

$\Rightarrow (6\lambda)^2 + (2\lambda)^2 + (3\lambda)^2 = 196$

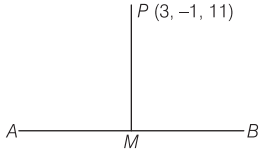
or $49\lambda^2 = 196$ or $\lambda^2 = 4$

or $\lambda = \pm 2$

Therefore, the required points are (14, 1, 5) and $(-10, -7, -7)$. The point nearer to the origin is (14, 1, 5).

D. Any point on line AB, $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$ [say]

is $M(2\lambda, 3\lambda + 2, 4\lambda + 3)$. Therefore, the direction ratios of PM are $2\lambda - 3$, $3\lambda + 3$ and $4\lambda - 8$.



But $PM \perp AB$

$\therefore \quad 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) = 0$

$\Rightarrow \quad 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$\Rightarrow \quad 29\lambda - 29 = 0$

$\therefore \quad \lambda = 1$

Therefore, foot of the perpendicular is $M(2, 5, 7)$.

VECTORS

A collection of best questions asked in JEE to revise your concepts

1. If P is any point on the circumcircle of ΔABC other than the vertices, H is the orthocentre of ΔABC, M is the mid-point of PH and D is the mid-point of BC, then [2016]

(a) DM is parallel to AP
(b) DM is perpendicular to AP
(c) AP is opposite to side DM
(d) None of the above

2. Let $ABCD$ be a tetrahedron with $AB = 41$, $BC = 36$, $CA = 7$, $DA = 18$, $DB = 27$ and $DC = 13$. Then, the square of the distance between the mid-points of AB and CD equals [2016]

(a) 1096 *(b)* 548 *(c)* 274 *(d)* 137

3. In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$ on $\sqrt{3}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is $\sqrt{3}$ and $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $|\alpha|$ is/are [2015]

(a) –1, 2 *(b)* 3, 4 *(c)* 4, 5 *(d)* 3

4. In R^2, let $\sqrt{3}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\hat{\mathbf{i}} + \sqrt{3}\,\hat{\mathbf{j}}$ and $\beta\hat{\mathbf{i}} + (1-\beta)\hat{\mathbf{j}}$ be the position vectors of X, Y and Z respectively with respect to the origin O. If the distance of Z from the bisector of the acute angle of $\mathbf{OX}$ with $\mathbf{OY}$ is $\frac{3}{\sqrt{2}}$, then the possible value(s) of $|\beta|$ is/are [2015]

(a) 8, 9 *(b)* 7, 8, 9 *(c)* 1, 2 *(d)* 3, 4

5. Let PQR be a triangle and $\mathbf{a} = \mathbf{QR}$, $\mathbf{b} = \mathbf{RP}$, $\mathbf{c} = \mathbf{PQ}$. If $|\mathbf{a}| = 12$, $|\mathbf{b}| = 4\sqrt{3}$ and $\mathbf{b}\cdot\mathbf{c} = 24$, then which of the following is/are true? [2015]

(a) $\frac{1}{2}|\mathbf{c}|^2 - |\mathbf{a}| = 12$ *(b)* $\frac{1}{2}|\mathbf{c}|^2 + |\mathbf{a}| = 30$

(c) $|\mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}| = 48\sqrt{3}$ *(d)* $\mathbf{a}\cdot\mathbf{b} = -72$

6. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\hat{\mathbf{a}}\times\hat{\mathbf{b}} + \hat{\mathbf{b}}\times\hat{\mathbf{c}} = p\hat{\mathbf{a}} + q\hat{\mathbf{b}} + r\hat{\mathbf{c}}$, where p, q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is [2014]

(a) 2 *(b)* 4 *(c)* 6 *(d)* 8

7. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\mathbf{a}$ is a non-zero vector perpendicular to $\mathbf{x}$ and $\mathbf{y}\times\mathbf{z}$ and $\mathbf{b}$ is a non-zero vector perpendicular to $\mathbf{y}$ and $\mathbf{z}\times\mathbf{x}$, then [2014]

(a) $\mathbf{b} = (\mathbf{b}\cdot\mathbf{z})(\mathbf{z} - \mathbf{x})$ *(b)* $\mathbf{a} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{y} - \mathbf{z})$
(c) $\mathbf{a}\cdot\mathbf{b} = -(\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})$ *(d)* $\mathbf{a} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{z} - \mathbf{y})$

8. Let $A_1, A_2, A_3, \ldots, A_n (n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin and $\mathbf{a}_k$ be the position vector of the point A_k; $k = 1, 2, 3, \ldots, n$.

If $\left|\sum_{k=1}^{n-1}(\mathbf{a}_k \times \mathbf{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\mathbf{a}_k \cdot \mathbf{a}_{k+1})\right|$, then the minimum value of n is [2014]

(a) 1 *(b)* 2 *(c)* 8 *(d)* 9

9. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors satisfying $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2 = 9$, then $|2\hat{\mathbf{a}} + 5\hat{\mathbf{b}} + 5\hat{\mathbf{c}}|$ is equal to [2012]

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

10. If the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$ and the area of the triangle with adjacent sides parallel to $\mathbf{a}$ and $\mathbf{b}$ is 3, then $\mathbf{a}\cdot\mathbf{b}$ equals [2013]

(a) $\sqrt{3}$ *(b)* $2\sqrt{3}$ *(c)* $4\sqrt{3}$ *(d)* $\frac{\sqrt{3}}{2}$

11. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{29}$ and $\mathbf{a}\times(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\times\mathbf{b}$, then the possible value of $(\mathbf{a} + \mathbf{b})(-7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ is [2012]

(a) 0 *(b)* 3 *(c)* 4 *(d)* 8

12. If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $((\mathbf{a}\times\mathbf{b})\times\mathbf{a})\cdot((\mathbf{b}\times\mathbf{a})\times\mathbf{b})$ equals [2015]

(a) $-(\mathbf{a}\cdot\mathbf{b})|\mathbf{a}\times\mathbf{b}|^2$ *(b)* $|\mathbf{a}\times\mathbf{b}|^2\,\mathbf{a}^2$
(c) $|\mathbf{a}\times\mathbf{b}|^2\,\mathbf{b}^2$ *(d)* $(\mathbf{a}\cdot\mathbf{b})\,|\mathbf{a}\times\mathbf{b}|^2$

13. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three coplanar unit vectors and a unit vector $\hat{\mathbf{d}}$ is perpendicular to them. If $(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times(\hat{\mathbf{c}}\times\hat{\mathbf{d}}) = \frac{1}{6}\hat{\mathbf{i}} - \frac{1}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$ and the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is 30°, then $\hat{\mathbf{c}}$ equals [2014]

(a) $\frac{1}{3}(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ *(b)* $\pm\frac{1}{3}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\frac{1}{3}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ *(d)* $\mp\frac{1}{3}(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

14. If $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = \lambda[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]^2$, then λ is equal to [2014]

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

15. Let $\mathbf{PR} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{SQ} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ be diagonals of a parallelogram $PQRS$ and $\mathbf{PT} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is another vector. Then, the volume of the parallelopiped determined by the vectors **PT**, **PQ** and **PS** is [2013]

(a) 5 *(b)* 20 *(c)* 10 *(d)* 30

16. Let $\mathbf{a} = -\hat{\mathbf{i}} - \hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ be three given vectors. If $\mathbf{r}$ is a vector such that $\mathbf{r}\times\mathbf{b} = \mathbf{c}\times\mathbf{b}$ and $\mathbf{r}\cdot\mathbf{a} = 0$, then the value of $\mathbf{r}\cdot\mathbf{b}$ is [2011]

(a) 4 *(b)* 8 *(c)* 6 *(d)* 9

17. Two adjacent sides of a parallelogram $ABCD$ are given by $\mathbf{AB} = 2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$ and $\mathbf{AD} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of angle α is given by [2010]

(a) $\frac{8}{9}$ *(b)* $\frac{\sqrt{17}}{9}$ *(c)* $\frac{1}{9}$ *(d)* $\frac{4\sqrt{5}}{9}$

18. If $\mathbf{a}$ and $\mathbf{b}$ are vectors in space given by $\mathbf{a} = \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}}}{\sqrt{5}}$ and $\mathbf{b} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{14}}$, then the value of $(2\mathbf{a} + \mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a} - 2\mathbf{b})]$ is [2010]

(a) 2 *(b)* 3 *(c)* 4 *(d)* 5

19. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$ be three vectors. A vector $\mathbf{r}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by [2011]

(a) $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *(b)* $-3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$

(c) $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ *(d)* $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

20. If $\mathbf{a} = \hat{\mathbf{j}} + \sqrt{3}\,\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{j}} + \sqrt{3}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 2\sqrt{3}\,\hat{\mathbf{k}}$ form a triangle, then the internal angle of the triangle between $\mathbf{a}$ and $\mathbf{b}$ is [2011]

(a) $\frac{\pi}{6}$ *(b)* $\frac{2\pi}{3}$

(c) $\frac{\pi}{3}$ *(d)* $\frac{\pi}{2}$

Answers with Explanation

1. ***(b)*** Let the origin O be the circumcentre of ΔABC and the position vectors of vertices A, B and C be $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Then, the position vector of its centroid G is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$. Since, the centroid G divides the segment joining circumcentre O and orthocentre H in the ratio 1 : 2. So, the position vector of H is $\mathbf{a}+\mathbf{b}+\mathbf{c}$. Let $\mathbf{r}$ be the position vector of point P on the circumcircle. Then, the position vector of M is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{r}}{2}$. The position vector of D is $\frac{\mathbf{b}+\mathbf{c}}{2}$.

$$\therefore \quad \mathbf{DM} = \frac{\mathbf{a}+\mathbf{r}}{2}$$

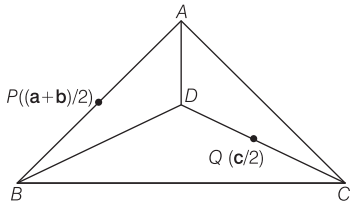
$$\text{Now,} \quad \mathbf{DM}\cdot\mathbf{AP} = \left(\frac{\mathbf{a}+\mathbf{r}}{2}\right)\cdot(\mathbf{r}-\mathbf{a})$$

$$= \frac{1}{2}(|\mathbf{r}|^2 - |\mathbf{a}|^2) = 0$$

Hence, $DM \perp AP$.

2. ***(d)*** Taking D as the origin, let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Then, $AB = 41$, $BC = 36$, $CA = 7$, $DA = 18$, $DB = 27$ and $DC = 13$.

$\Rightarrow |\mathbf{b}-\mathbf{a}| = 41, |\mathbf{c}-\mathbf{b}| = 36, |\mathbf{a}-\mathbf{c}| = 7, |\mathbf{a}| = 18, |\mathbf{b}| = 27$ and $|\mathbf{c}| = 13$



Let P and Q be the mid-points of AB and CD, respectively. Then, the position vectors of P and Q are $\frac{\mathbf{a}+\mathbf{b}}{2}$ and $\frac{\mathbf{c}}{2}$, respectively.

$$\therefore \quad PQ = |\mathbf{PQ}| = \left|\frac{\mathbf{c}}{2} - \frac{\mathbf{a}+\mathbf{b}}{2}\right|$$

$$= \frac{1}{2}|\mathbf{a}+\mathbf{b}-\mathbf{c}|$$

$$\text{Now, } |\mathbf{a}+\mathbf{b}-\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) - 2(\mathbf{b}\cdot\mathbf{c}) - 2(\mathbf{c}\cdot\mathbf{a})$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + \{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\} - \{|\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{b}-\mathbf{c}|^2\} - \{|\mathbf{c}|^2 + |\mathbf{a}|^2 - |\mathbf{c}-\mathbf{a}|^2\}$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2 - |\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2$$

$$= 324 + 729 - 169 - 1681 + 1296 + 49 = 548$$

$$\therefore \quad PQ^2 = \frac{1}{4}|\mathbf{a}+\mathbf{b}-\mathbf{c}|^2 = 137$$

3. ***(a)*** Let $\mathbf{a} = \alpha\hat{\mathbf{i}} + \beta\hat{\mathbf{j}}$ and $\mathbf{b} = \sqrt{3}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$. It is given that the magnitude of projection of $\mathbf{a}$ on $\mathbf{b}$ is $\sqrt{3}$.

$$\therefore \quad \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right| = \sqrt{3}$$

$$\Rightarrow \quad \left|\frac{\sqrt{3}\alpha + \beta}{2}\right| = \sqrt{3}$$

$$\Rightarrow \quad \sqrt{3}\alpha + \beta = \pm 2\sqrt{3}$$

$$\Rightarrow \quad \sqrt{3}(2 + \sqrt{3}\beta) + \beta = \pm 2\sqrt{3} \qquad [\because \alpha = 2 + \sqrt{3}\,\beta]$$

$$\Rightarrow \quad 2\sqrt{3} + 4\beta = \pm 2\sqrt{3}$$

$$\Rightarrow \quad \beta = 0 \text{ or } \beta = -\sqrt{3}$$

On putting these values in $\alpha = 2 + \sqrt{3}\beta$, we get

$$\alpha = 2 \quad \text{or} \quad \alpha = -1$$

4. ***(c)*** Clearly, coordinates of X, Y and Z are $(\sqrt{3}, 1), (1, \sqrt{3})$ and $(\beta, 1-\beta)$ respectively. We observe that OX and OY make angles of $30°$ and $60°$, respectively with X-axis. So, the bisector of acute angle between them makes $45°$ angle with X-axis and hence its equation is $x = y$ or $x - y = 0$. It is given that the distance of $Z(\beta, 1-\beta)$ from $x - y = 0$ is $\frac{3}{\sqrt{2}}$.

$$\therefore \quad \left|\frac{\beta - (1-\beta)}{\sqrt{(1)^2 + (-1)^2}}\right| = \frac{3}{\sqrt{2}}$$

$$\Rightarrow \quad |2\beta - 1| = 3$$

$$\Rightarrow \quad 2\beta - 1 = \pm 3$$

$$\Rightarrow \quad \beta = 2, -1$$

Hence, $\quad |\beta| = 1, 2$

5. ***(a, c, d)*** We have,

$$\mathbf{a} = \mathbf{QR},\ \mathbf{b} = \mathbf{RP} \text{ and } \mathbf{c} = \mathbf{PQ}$$

$$\Rightarrow \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{QR} + \mathbf{RP} + \mathbf{PQ}$$

$$\Rightarrow \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{QQ}$$

$$\Rightarrow \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

$$\Rightarrow \quad \mathbf{b} + \mathbf{c} = -\mathbf{a} \quad \Rightarrow \quad |\mathbf{b} + \mathbf{c}| = |-\mathbf{a}|$$

$$\Rightarrow \quad |\mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2$$

$$\Rightarrow \quad |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{b}\cdot\mathbf{c}) = |\mathbf{a}|^2$$

$$\Rightarrow \quad 48 + |\mathbf{c}|^2 + 48 = 144$$

$$\Rightarrow \quad |\mathbf{c}| = 4\sqrt{3}$$

$$\therefore \quad \frac{1}{2}|\mathbf{c}|^2 - |\mathbf{a}| = 24 - 12 = 12$$

and $\quad \frac{1}{2}|\mathbf{c}|^2 + |\mathbf{a}| = 24 + 12 = 36$

Again, $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$$\Rightarrow \quad |\mathbf{a} + \mathbf{b}| = |-\mathbf{c}|$$

$$\Rightarrow \quad |\mathbf{a} + \mathbf{b}|^2 = |-\mathbf{c}|^2$$

$$\Rightarrow \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) = |\mathbf{c}|^2$$

$$\Rightarrow \quad 144 + 48 + 2(\mathbf{a}\cdot\mathbf{b}) = 48$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = -72$$

Again, $\quad \mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

$$\Rightarrow \quad \mathbf{a}\times(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a}\times 0$$

$$\Rightarrow \quad \mathbf{a}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} = 0$$

$$\Rightarrow \quad \mathbf{a}\times\mathbf{b} = \mathbf{c}\times\mathbf{a}$$

$$\therefore \quad |\mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a}| = |2(\mathbf{a}\times\mathbf{b})| = 2|\mathbf{a}\times\mathbf{b}|$$

$$= 2\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2}$$

$$= 2\sqrt{144\times 48 - (-72)^2}$$

$$= 48\sqrt{3}$$

6. ***(b)*** We have,

$$\hat{\mathbf{a}}\times\hat{\mathbf{b}} + \hat{\mathbf{b}}\times\hat{\mathbf{c}} = p\hat{\mathbf{a}} + q\hat{\mathbf{b}} + r\hat{\mathbf{c}} \qquad \ldots(i)$$

Taking dot product with $\hat{\mathbf{a}}$, we get

$$\hat{\mathbf{a}}\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{b}}) + \hat{\mathbf{a}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}}) = p(\hat{\mathbf{a}}\cdot\hat{\mathbf{a}}) + q(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) + r(\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})$$

$$\Rightarrow \quad [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}] = p + \frac{q}{2} + \frac{r}{2} \qquad \ldots(ii)$$

Taking dot product on both sides of Eq. (i) by $\hat{\mathbf{b}}$, we get

$$\hat{\mathbf{b}}\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{b}}) + \hat{\mathbf{b}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}}) = p(\hat{\mathbf{b}}\cdot\hat{\mathbf{a}}) + q(\hat{\mathbf{b}}\cdot\hat{\mathbf{b}}) + r(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}})$$

$$\Rightarrow \quad 0 = \frac{p}{2} + q + \frac{r}{2} \qquad \ldots(iii)$$

Taking dot product on both sides of Eq. (i) by $\hat{\mathbf{c}}$, we get

$$\hat{\mathbf{c}}\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{b}}) + \hat{\mathbf{c}}\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{c}}) = p(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}}) + q(\hat{\mathbf{c}}\cdot\hat{\mathbf{b}}) + r(\hat{\mathbf{c}}\cdot\hat{\mathbf{c}})$$

$$\Rightarrow \quad [\hat{\mathbf{c}}\ \hat{\mathbf{a}}\ \hat{\mathbf{b}}] = \frac{p}{2} + \frac{q}{2} + r$$

$$\Rightarrow \quad [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}] = \frac{p}{2} + \frac{q}{2} + r \qquad \ldots(iv)$$

From Eqs. (ii) and (iv), we get

$$p + \frac{q}{2} + \frac{r}{2} = \frac{p}{2} + \frac{q}{2} + r \Rightarrow p = r$$

On putting $p = r$ in Eq. (iii), we get $q = -r$.

Thus, $\quad p = r = -q$

$$\therefore \quad \frac{p^2 + 2q^2 + r^2}{q^2} = \frac{(-q)^2 + 2q^2 + (-q)^2}{q^2} = 4$$

7. ***(a, b, c)*** We have, $|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = \sqrt{2}$

Now, $\quad \mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\frac{\pi}{3} = 1$

$$\mathbf{y}\cdot\mathbf{z} = 1$$

and $\quad \mathbf{z}\cdot\mathbf{x} = 1$

It is given that $\mathbf{a}$ is perpendicular to $\mathbf{x}$ and $\mathbf{y}\times\mathbf{z}$ and $\mathbf{b}$ is perpendicular to $\mathbf{y}$ and $\mathbf{z}\times\mathbf{x}$.

$$\therefore \quad \mathbf{a} \parallel \mathbf{x}\times(\mathbf{y}\times\mathbf{z}) \text{ and } \mathbf{b} \parallel \mathbf{y}\times(\mathbf{z}\times\mathbf{x})$$

$$\Rightarrow \quad \mathbf{a} = \lambda_1\{\mathbf{x}\times(\mathbf{y}\times\mathbf{z})\}$$

and $\quad \mathbf{b} = \lambda_2\{\mathbf{y}\times(\mathbf{z}\times\mathbf{x})\}$

$$\Rightarrow \quad \mathbf{a} = \lambda_1\{(\mathbf{x}\cdot\mathbf{z})\mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{z}\}$$

and $\quad \mathbf{b} = \lambda_2\{(\mathbf{y}\cdot\mathbf{x})\mathbf{z} - (\mathbf{y}\cdot\mathbf{z})\mathbf{x}\}$

$$\Rightarrow \quad \mathbf{a} = \lambda_1(\mathbf{y} - \mathbf{z}) \text{ and } \mathbf{b} = \lambda_2(\mathbf{z} - \mathbf{x}) \qquad \ldots(i)$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{y} = \lambda_1(\mathbf{y}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{z})$$

and $\quad \mathbf{b}\cdot\mathbf{z} = \lambda_2(\mathbf{z}\cdot\mathbf{z} - \mathbf{x}\cdot\mathbf{z})$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{y} = \lambda_1(2 - 1)$$

and $\quad \mathbf{b}\cdot\mathbf{z} = \lambda_2(2 - 1)$

$$\Rightarrow \quad \lambda_1 = \mathbf{a}\cdot\mathbf{y} \text{ and } \lambda_2 = \mathbf{b}\cdot\mathbf{z}$$

On putting the values of λ_1 and λ_2 in Eq. (i), we get

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{y} - \mathbf{z}) \text{ and } \mathbf{b} = (\mathbf{b}\cdot\mathbf{z})(\mathbf{z} - \mathbf{x})$$

Now, $\quad \mathbf{a}\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{y} - \mathbf{z})\cdot(\mathbf{b}\cdot\mathbf{z})(\mathbf{z} - \mathbf{x})$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})\{(\mathbf{y} - \mathbf{z})\cdot(\mathbf{z} - \mathbf{x})\}$$

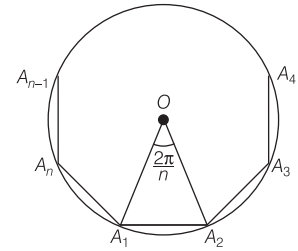
$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = \{(\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})\}(\mathbf{y}\cdot\mathbf{z} - \mathbf{y}\cdot\mathbf{x} - \mathbf{z}\cdot\mathbf{z} + \mathbf{z}\cdot\mathbf{x})$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})(1 - 1 - 2 + 1)$$

$$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = -(\mathbf{a}\cdot\mathbf{y})(\mathbf{b}\cdot\mathbf{z})$$

8. ***(c)*** Let $OA_1 = OA_2 = \ldots = OA_n = r$.

Clearly, $\angle A_{k-1}OA_k = \frac{2\pi}{n}$ for $k = 2, 3, 4, \ldots, n$



Now, $\quad \left|\sum_{k=1}^{n-1}(\mathbf{a}_k \times \mathbf{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\mathbf{a}_k \cdot \mathbf{a}_{k+1})\right|$

$$\Rightarrow \quad \sum_{k=1}^{n-1}|\mathbf{a}_k||\mathbf{a}_{k+1}|\sin\frac{2\pi}{n} = \sum_{k=1}^{n-1}|\mathbf{a}_k||\mathbf{a}_{k+1}|\cos\frac{2\pi}{n}$$

$$\Rightarrow \quad \sum_{k=1}^{n-1} r^2\sin\frac{2\pi}{n} = \sum_{k=1}^{n-1} r^2\cos\frac{2\pi}{n}$$

$$\Rightarrow \quad r^2(n-1)\sin\frac{2\pi}{n} = r^2(n-1)\cos\frac{2\pi}{n}$$

$$\Rightarrow \quad \tan\frac{2\pi}{n} = 1 \Rightarrow \frac{2\pi}{n} = \frac{\pi}{4} \Rightarrow n = 8$$

9. ***(d)*** We know that,

$$|\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}|^2 = |\hat{\mathbf{a}}|^2 + |\hat{\mathbf{b}}|^2 + |\hat{\mathbf{c}}|^2 + 2(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} + \hat{\mathbf{b}}\cdot\hat{\mathbf{c}} + \hat{\mathbf{c}}\cdot\hat{\mathbf{a}})$$

and $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2 = 2(|\hat{\mathbf{a}}|^2 + |\hat{\mathbf{b}}|^2 + |\hat{\mathbf{c}}|^2) - 2(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}} + \hat{\mathbf{b}}\cdot\hat{\mathbf{c}} + \hat{\mathbf{c}}\cdot\hat{\mathbf{a}})$

$$\therefore |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2 = 3\{|\hat{\mathbf{a}}|^2 + |\hat{\mathbf{b}}|^2 + |\hat{\mathbf{c}}|^2\} - |\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}|^2$$

$$\Rightarrow \quad 9 = 3\times 3 - |\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}|^2$$

$$\Rightarrow \quad |\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}| = 0$$

$$\Rightarrow \quad \hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}} = 0$$

$$\Rightarrow \quad \hat{\mathbf{b}} + \hat{\mathbf{c}} = -\hat{\mathbf{a}}$$

$$\therefore \quad |2\hat{\mathbf{a}} + 5\hat{\mathbf{b}} + 5\hat{\mathbf{c}}| = |2\hat{\mathbf{a}} - 5\hat{\mathbf{a}}| = 3|\hat{\mathbf{a}}| = 3$$

10. ***(b)*** We have, $\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = 3$

$$\Rightarrow \quad |\mathbf{a}\times\mathbf{b}| = 6$$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{3} = 6$$

$$\Rightarrow \quad |\mathbf{a}||\mathbf{b}| = 4\sqrt{3}$$

$$\therefore \quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{3} = 4\sqrt{3}\times\frac{1}{2} = 2\sqrt{3}$$

11. ***(d)*** We have, $\mathbf{a}\times(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\times\mathbf{b}$

$$\Rightarrow \quad \mathbf{a}\times(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) + \mathbf{b}\times(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad (\mathbf{a} + \mathbf{b})\times(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad (\mathbf{a} + \mathbf{b}) \,||\, (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$\Rightarrow \quad \mathbf{a} + \mathbf{b} = \pm(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \qquad [\because |\mathbf{a} + \mathbf{b}| = \sqrt{29}]$$

$$\therefore (\mathbf{a} + \mathbf{b})\cdot(-7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = \pm(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})\cdot(-7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = \pm 8$$

12. ***(a)*** $((\mathbf{a}\times\mathbf{b})\times\mathbf{a})\cdot((\mathbf{b}\times\mathbf{a})\times\mathbf{b})$

$$= (\mathbf{a}\times(\mathbf{a}\times\mathbf{b}))\cdot(\mathbf{b}\times(\mathbf{b}\times\mathbf{a}))$$

$$= ((\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b})\cdot((\mathbf{b}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{b})\mathbf{a})$$

$$= (\mathbf{a}\cdot\mathbf{b})^3 - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a})$$

$$= (\mathbf{a}\cdot\mathbf{b})^3 - |\mathbf{a}|^2|\mathbf{b}|^2(\mathbf{a}\cdot\mathbf{b}) - |\mathbf{a}|^2|\mathbf{b}|^2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{a}|^2|\mathbf{b}|^2(\mathbf{a}\cdot\mathbf{b})$$

$$= (\mathbf{a}\cdot\mathbf{b})\{(\mathbf{a}\cdot\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2\}$$

$$= -(\mathbf{a}\cdot\mathbf{b})|\mathbf{a}\times\mathbf{b}|^2$$

13. ***(d)*** Clearly, $(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times(\hat{\mathbf{c}}\times\hat{\mathbf{d}})$

$$= \{(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{d}}\}\hat{\mathbf{c}} - \{(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{c}}\}\hat{\mathbf{d}}$$

$$= [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{d}}]\hat{\mathbf{c}} - [\hat{\mathbf{a}}\ \hat{\mathbf{b}}\ \hat{\mathbf{c}}]\hat{\mathbf{d}}$$

$$= \{(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{d}}\}\hat{\mathbf{c}} - 0\cdot\hat{\mathbf{d}}$$

$$= \pm|(\hat{\mathbf{a}}\times\hat{\mathbf{b}}||\hat{\mathbf{d}})|\hat{\mathbf{c}} \quad [\because (\hat{\mathbf{a}}\times\hat{\mathbf{b}})\cdot\hat{\mathbf{d}} = \pm|(\hat{\mathbf{a}}\times\hat{\mathbf{b}}||\hat{\mathbf{d}})|]$$

$$= (\pm|\hat{\mathbf{a}}||\hat{\mathbf{b}}|\sin 30^\circ)\hat{\mathbf{c}} \qquad [\because |\hat{\mathbf{d}}| = 1]$$

$$= \pm\frac{1}{2}\hat{\mathbf{c}}$$

$$\therefore \quad \frac{1}{6}\hat{\mathbf{i}} - \frac{1}{3}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}} = \pm\frac{1}{2}\mathbf{c}$$

$$\Rightarrow \quad \hat{\mathbf{c}} = \pm\frac{1}{3}(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

14. ***(b)*** We know that,

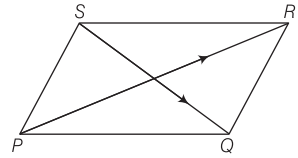
$$[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

Therefore,

$$[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = \lambda[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

$$\therefore \quad \lambda = 1$$

15. ***(c)*** In ΔPQR and ΔPSQ, we have



$$\mathbf{PQ} + \mathbf{QR} = \mathbf{PR} \text{ and } \mathbf{PS} + \mathbf{SQ} = \mathbf{PQ}$$

$$\Rightarrow \quad \mathbf{PQ} + \mathbf{QR} = \mathbf{PR} \text{ and } \mathbf{PQ} - \mathbf{PS} = \mathbf{SQ}$$

$$\Rightarrow \quad \mathbf{PQ} + \mathbf{PS} = \mathbf{PR} \text{ and } \mathbf{PQ} - \mathbf{PS} = \mathbf{SQ}$$

$$\Rightarrow \quad \mathbf{PQ} = \frac{\mathbf{PR} + \mathbf{SQ}}{2} \text{ and } \mathbf{PS} = \frac{\mathbf{PR} - \mathbf{SQ}}{2}$$

$$\Rightarrow \quad \mathbf{PQ} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}} \text{ and } \mathbf{PS} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Thus, $\mathbf{PQ} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $\mathbf{PS} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

and $\mathbf{PT} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\therefore [\mathbf{PT}\ \ \mathbf{PQ}\ \ \mathbf{PS}] = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 10$$

Hence, volume of the parallelopiped determined by the vectors **PT**, **PQ** and **PS** is 10 cu units.

16. ***(d)*** We have, $\mathbf{r}\times\mathbf{b} = \mathbf{c}\times\mathbf{b}$

$$\Rightarrow \quad (\mathbf{r} - \mathbf{c})\times\mathbf{b} = 0$$

$$\Rightarrow \quad (\mathbf{r} - \mathbf{c}) \,||\, \mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} - \mathbf{c} = \lambda\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r} = \mathbf{c} + \lambda\mathbf{b} \qquad \ldots(i)$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{a} = \mathbf{c}\cdot\mathbf{a} + \lambda(\mathbf{b}\cdot\mathbf{a})$$

$$\Rightarrow \quad 0 = \mathbf{c}\cdot\mathbf{a} + \lambda(\mathbf{b}\cdot\mathbf{a}) \qquad [\because \mathbf{r}\cdot\mathbf{a} = 0]$$

$$\Rightarrow \quad \lambda = -\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}$$

On putting the value of λ in Eq. (i), we get

$$\mathbf{r} = \mathbf{c} - \frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\mathbf{b}$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{b} = (\mathbf{c}\cdot\mathbf{b}) - \frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}(\mathbf{b}\cdot\mathbf{b})$$

$$\Rightarrow \quad \mathbf{r}\cdot\mathbf{b} = 1 - \frac{(-4)}{1}\times 2 = 9$$

To be Continued at Page 86

BRAIN TEASERS

To boost up your rank in JEE Main and Advanced

1. Find the value of the series

$$2+\frac{5}{2!\,3}+\frac{5\cdot 7}{3!\,3^2}+\frac{5\cdot 7\cdot 9}{4!\,3^3}+\ldots .$$

2. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then prove that

$$\frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}=\frac{(x+y+z)^2+(a+b+c)^2}{x+y+z+a+b+c}.$$

3. Solve $6x^4-25x^3+12x^2+25x+6=0$.

4. If $b=\sqrt{3}, c=1$ and $A=30^\circ$, solve the triangle.

5. If $\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}=P+Q\cos\frac{\pi}{7}+R\cos\frac{2\pi}{7}$, where P, Q and R are non-zero constants, then find the value of $P+Q+R$.

6. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

7. If $f(x)-f(4020-x)=0$ and $\int_{2009}^{2011} x\,f(x)\,dx=\lambda\int_{2011}^{2009} f(x)\,dx$, then find the value of λ.

8. Solve the equation $5\cos\theta-2\sin\theta=2$, given that $\tan 23^\circ 12'=\frac{3}{7}$.

9. What is represented by the equation $(x^2-a^2)^2+(y^2-a^2)^2=a^4$?

10. Find the square root of $\frac{3}{2}(x-1)+\sqrt{2x^2-7x-4}$.

Solutions

1. We have,

$$2+\frac{5}{2!\,3}+\frac{5\cdot 7}{3!\,3^2}+\frac{5\cdot 7\cdot 9}{4!\,3^3}+\ldots$$

$$=2+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\frac{2^2}{3^2}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\frac{2^3}{3^3}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4!}\cdot\frac{2^4}{3^4}+\ldots$$

$$=1+\frac{\frac{3}{2}}{1}\cdot\left(\frac{2}{3}\right)^1+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\left(\frac{2}{3}\right)^3+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4!}\cdot\left(\frac{2}{3}\right)^4+\ldots$$

$$=\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=\left(\frac{1}{3}\right)^{-\frac{3}{2}}=3^{\frac{3}{2}}=3\sqrt{3}$$

2. Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$

Then, $x=ak, y=bk, z=ck$

Now, $\frac{x^2+a^2}{x+a}=\frac{a^2k^2+a^2}{ak+a}=a^2\frac{(k^2+1)}{a(k+1)}=\frac{a(k^2+1)}{k+1}$

$$\therefore\ \frac{x^2+a^2}{x+a}+\frac{y^2+b^2}{y+b}+\frac{z^2+c^2}{z+c}$$

$$=\frac{(k^2+1)a}{k+1}+\frac{(k^2+1)b}{k+1}+\frac{(k^2+1)c}{k+1}$$

$$=\frac{(k^2+1)(a+b+c)}{(k+1)}=\frac{k^2(a+b+c)^2+(a+b+c)^2}{k(a+b+c)+(a+b+c)}$$

$$=\frac{(ka+kb+kc)^2+(a+b+c)^2}{(ka+kb+kc)+(a+b+c)}$$

$$=\frac{(x+y+z)^2+(a+b+c)^2}{(x+y+z)+(a+b+c)}$$

Hence proved.

3. We have, $6x^4-25x^3+12x^2+25x+6=0$

$$\Rightarrow\quad 6x^2-25x+12+\frac{25}{x}+\frac{6}{x^2}=0$$

$$\Rightarrow \quad 6\left(x^2+\frac{1}{x^2}\right)-25\left(x-\frac{1}{x}\right)+12=0$$

$$\Rightarrow \quad 6\left(x-\frac{1}{x}\right)^2-25\left(x-\frac{1}{x}\right)+24=0$$

Let $x-\frac{1}{x}=y$

$\therefore \quad 6y^2-25y+24=0$

$\Rightarrow \quad (2y-3)(3y-8)=0$

$\Rightarrow \quad y=\frac{3}{2},\ \frac{8}{3}$

$\therefore \quad x-\frac{1}{x}=\frac{3}{2}$ or $x-\frac{1}{x}=\frac{8}{3}$

$\Rightarrow \quad \frac{x^2-1}{x}=\frac{3}{2}$ or $\frac{x^2-1}{x}=\frac{8}{3}$

$\Rightarrow \quad 2x^2-3x-2=0$

or $\quad 3x^2-8x-3=0$

$\Rightarrow \quad (2x+1)(x-2)=0$

or $\quad (x-3)(3x+1)=0$

$\Rightarrow \quad x=2,-\frac{1}{2},3,-\frac{1}{3}$

4. We know that,

$$\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\cot 15^\circ$$

$$=\tan 15^\circ\cot 15^\circ \qquad \left[\because \tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}\right]$$

$$=1$$

$\therefore \quad \frac{B-C}{2}=45^\circ$

Also, $\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-15^\circ=75^\circ$

$\therefore \quad B=120^\circ$ and $C=30^\circ$

Since $A=C$

$\therefore \quad a=c=1$

5. We have, $\sin^2\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}$

$$=\left(\frac{1-\cos\frac{2\pi}{7}}{2}\right)\left(\frac{\cos\frac{2\pi}{7}-\cos\frac{6\pi}{7}}{2}\right)$$

$$=\frac{1}{4}\left(1-\cos\frac{2\pi}{7}\right)\left(\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{4}\left(\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}-\cos^2\frac{2\pi}{7}-\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{4}\left(\cos\frac{2\pi}{7}+\cos\frac{\pi}{7}-\frac{1}{2}-\frac{1}{2}\cos\frac{4\pi}{7}-\frac{1}{2}\cos\frac{3\pi}{7}-\frac{1}{2}\cos\frac{\pi}{7}\right)$$

$$=\frac{1}{4}\left(\cos\frac{2\pi}{7}-\frac{1}{2}\cos\frac{\pi}{7}-\frac{1}{2}\right)$$

$\therefore \quad P+Q+R=\left(-\frac{1}{8}\right)+\left(-\frac{1}{8}\right)+\frac{1}{4}=0$

6. Let x, y and z be three distinct positive integers satisfying the given conditions. We may assume that, $x<y<z$. Thus, we have

$$\frac{1}{y}+\frac{1}{z}=\frac{a}{x},\quad \frac{1}{z}+\frac{1}{x}=\frac{b}{y},\quad \frac{1}{x}+\frac{1}{y}=\frac{c}{z}$$

for some positive integers, a, b and c.

Thus, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{a+1}{x}=\frac{b+1}{y}=\frac{c+1}{z}=r$ [say]

Since $x<y<z$, we observe that $a<b<c$, which gives

$$r=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{r}{a+1}+\frac{r}{b+1}+\frac{r}{c+1}$$

$$\Rightarrow \quad \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=1 \qquad \ldots(i)$$

Using $a<b<c$, we get

$$1=\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}<\frac{3}{a+1}$$

Thus, $a<2$

We conclude that $a=1$.

On putting the value of a in Eq. (i), we get

$$\frac{1}{b+1}+\frac{1}{c+1}=1-\frac{1}{2}=\frac{1}{2}$$

Since $b<c$ gives $\frac{1}{2}<\frac{2}{b+1}$

Thus, $b+1<4$ or $b<3$.

Since $b>a=1$, we must have $b=2$, which gives

$$\frac{1}{c+1}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

or $\quad c=5$

Thus, $\quad x:y:z=a+1:b+1:c+1=2:3:6$

Thus, the required numbers with the least sum are 2, 3, 6.

7. Let $I=\int_{2009}^{2011} x\ f(x)\ dx$

$$\Rightarrow \quad I=\int_{2009}^{2011}(2009+2011-x)\ f(2009+2011-x)\ dx$$

$$\Rightarrow \quad I=\int_{2009}^{2011}(4020-x)\ f(4020-x)\ dx$$

$$\Rightarrow \quad I=\int_{2009}^{2011}(4020-x)\ f(x)\ dx$$

$$\Rightarrow \quad I=4020\int_{2009}^{2011} f(x)\ dx-\int_{2009}^{2011} x\ f(x)\ dx$$

$$\Rightarrow \quad I=4020\int_{2009}^{2011} f(x)\ dx-I$$

$$\Rightarrow 2I=4020\int_{2009}^{2011} f(x)\ dx$$

$$\Rightarrow \quad I=2010\int_{2009}^{2011} f(x)\ dx$$

$$\Rightarrow \quad I=-2010\int_{2011}^{2009} f(x)dx$$

$\therefore \quad \lambda=-2010$

8. Let $t=\tan\frac{\theta}{2}$

Then, $\sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{2t}{1+t^2}$

and $\cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \dfrac{1-t^2}{1+t^2}$

Now, the equation becomes

$$5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$$

$\Rightarrow \quad 7t^2 + 4t - 3 = 0$

$\Rightarrow \quad (7t-3)(t+1) = 0$

$\Rightarrow \quad t = -1, \dfrac{3}{7}$

$\therefore \quad \tan\dfrac{\theta}{2} = \tan(-45°)$ or $\tan 23°12'$

$\Rightarrow \quad \dfrac{\theta}{2} = n\pi - \dfrac{\pi}{4}$ or $n\pi + 23°12'$

$\therefore \quad \theta = 2n\pi - \dfrac{\pi}{2}$ or $2n\pi + 46°24'$

9. Given equation is $(x^2-a^2)^2 + (y^2-a^2)^2 = a^4$

$\Rightarrow \quad x^4 + y^4 - 2a^2(x^2+y^2) + a^4 = 0$

$\Rightarrow \quad (x^2+y^2)^2 - 2a^2(x^2+y^2) + a^4 - 2x^2y^2 = 0$

$\Rightarrow \quad (x^2+y^2-a^2)^2 - (\sqrt{2}xy)^2 = 0$

$\Rightarrow \quad (x^2+\sqrt{2}xy+y^2-a^2)(x^2-\sqrt{2}xy+y^2-a^2) = 0$

Therefore, the locus consists of the two ellipses

$x^2+\sqrt{2}xy+y^2-a^2 = 0$

and $x^2-\sqrt{2}xy+y^2-a^2 = 0$

These ellipses are equal and their semi-axes would found to be $a\sqrt{2+\sqrt{2}}$ and $a\sqrt{2-\sqrt{2}}$.

The major axis of the first is inclined at an angle of 135° to X-axis and that of the second at an angle of 45°.

10. Given expression

$$= \frac{3}{2}(x-1) + \sqrt{2x^2-7x-4}$$

$$= \frac{1}{2}\{3(x-1) + 2\sqrt{(2x+1)(x-4)}\}$$

$$= \frac{1}{2}\{(2x+1) + (x-4) + 2\sqrt{(2x+1)(x-4)}\}$$

$$= \frac{1}{2}\{\sqrt{2x+1} + \sqrt{x-4}\}^2$$

$\therefore$ The square root of the given expression

$$= \frac{\sqrt{2x+1}+\sqrt{x-4}}{\sqrt{2}}$$

Continued from Page 83

17. ***(b)*** Clearly, AD' is perpendicular to **AB** and lies in the plane of **AB** and **ADB**.

$\therefore \quad \mathbf{AD'} = \mathbf{AB} \times (\mathbf{AB} \times \mathbf{AD})$

$\Rightarrow \quad \mathbf{AD'} = (\mathbf{AB}\cdot\mathbf{AD})\mathbf{AB} - (\mathbf{AB}\cdot\mathbf{AB})\mathbf{AD}$

$\Rightarrow \quad \mathbf{AD'} = 5(61\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$

$\therefore \quad \cos\alpha = \dfrac{|\mathbf{AD}\cdot\mathbf{AD'}|}{|\mathbf{AD}||\mathbf{AD'}|}$

$\Rightarrow \quad \cos\alpha = \dfrac{(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})\cdot 5(61\hat{\mathbf{i}} - 10\hat{\mathbf{j}} - 2\hat{\mathbf{k}})}{5\sqrt{1^2+2^2+2^2}\sqrt{61^2+10^2+2^2}}$

$= \dfrac{\sqrt{17}}{9}$

18. ***(d)*** Here, $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b})$

$= (\mathbf{a}\times\mathbf{b})\times\mathbf{a} - 2(\mathbf{a}\times\mathbf{b})\times\mathbf{b}$

$= -\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) + 2\mathbf{b}\times(\mathbf{a}\times\mathbf{b})$

$= -\{(\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b}\}$

$\quad + 2\{(\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}\}$

$= 2\mathbf{a} + \mathbf{b} \quad [\because \mathbf{a}\cdot\mathbf{b} = 0, \mathbf{a}\cdot\mathbf{a} = 1, \mathbf{b}\cdot\mathbf{b} = 1]$

$\therefore (2\mathbf{a}+\mathbf{b})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}-2\mathbf{b})]$

$= (2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$

$= 4|\mathbf{a}|^2 + |\mathbf{b}|^2 + 4(\mathbf{a}\cdot\mathbf{b})$

$= 4 + 1 + 0 = 5$

19. ***(c)*** Let $\mathbf{r}$ be the required vector.

Then, $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$

$\Rightarrow \quad \mathbf{r} = (x+y)\hat{\mathbf{i}} + (x-y)\hat{\mathbf{j}} + (x+y)\hat{\mathbf{k}}$

Since, the projection of $\mathbf{r}$ on $\mathbf{c}$ is $\dfrac{1}{\sqrt{3}}$.

$\therefore \quad \mathbf{r}\cdot\dfrac{1}{\sqrt{3}}(\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}) = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad x + y - (x-y) - (x+y) = 1$

$\Rightarrow \quad -x + y = 1$

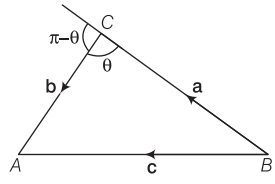
$\Rightarrow \quad y = x + 1$

$\therefore \mathbf{r} = (2x+1)\hat{\mathbf{i}} - \hat{\mathbf{j}} + (2x+1)\hat{\mathbf{k}}, x \in R$

For $x = 1$, we get $\mathbf{r} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

20. ***(b)*** Clearly, $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $|\mathbf{a}| = |\mathbf{b}| = 2$

So, **a**, **b**, **c** form an isosceles triangle as shown below.



Let θ be the internal angle between **a** and **b**.

Then, $\cos(\pi-\theta) = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$\Rightarrow \quad -\cos\theta = \dfrac{-1+3}{2\times 2} = \dfrac{1}{2}$

$\Rightarrow \quad \cos\theta = \dfrac{-1}{2}$

$\therefore \quad \theta = \dfrac{2\pi}{3}$

TARGET CBSE 2017

TEST DRIVE

Comprehensive Simulator Test Series for XII Board Exam.

BOARD EXAM

(PAPER ONE PREP UP)

Typical Questions with Crispy Solutions

FULL TEST 3

(COMPLETE SYLLABUS) WITH SOLUTIONS

INSTRUCTIONS

- Question Numbers 1-4 in **Section A** carrying 1 mark each.
- Question Numbers 5-12 in **Section B** carrying 2 marks each.
- Question Numbers 13-23 in **Section C** carrying 4 marks each.
- Question Numbers 24-29 in **Section D** carrying 6 marks each.

Section A

1. Using principal values, evaluate $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$.

[Delhi 2009C]

2. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = \sin x$ and $g(x) = 5x^2$, then find $gof\,(x)$.

3. Find x from the matrix equation $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

[Foreign 2010]

4. Find $|\mathbf{a} \times \mathbf{b}|$, if $\mathbf{a} = \hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.

[Delhi 2008C]

Section B

5. Solve for x, if $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$, $0 < x < 1$.

6. Solve the matrix equation $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, where X is 2×2 matrix.

7. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

8. Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

9. Evaluate $\int \frac{x^3}{1 + x^8}\,dx$.

10. Show that the function f defined by $f(x) = |1 - x + |x||$ is a continuous function.

11. Prove that the three points $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $-7\mathbf{b} + 10\mathbf{c}$ are collinear.

12. Two dice are thrown. Find the probability of getting an odd number on the first and a multiple of 3 on the other.

Section C

13. Show that $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $x^2 - 6x + 17 = 0$. Hence, find A^{-1}.

14. Evaluate $\int \sin x \sin 2x \sin 3x\,dx$.

[Delhi 2012]

Or

Evaluate $\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx$.

15. Differentiate $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right)$ with respect to x.

16. Evaluate $\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$.

[Delhi 2011; All India 2010]

17. Form the differential equation representing the family of ellipses having foci on X-axis and centre at the origin.

[Delhi 2009C]

18. Find the vector equation of a line passing through the point with position vector $(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$ and perpendicular to the plane $\mathbf{r} \cdot (6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + 2 = 0$. Also, find the point of intersection of this line and the plane.

19. In a group of 50 scouts in a camp, 30 are well trained in first aid techniques while the remaining are well trained in hospitality but not in first aid. Two scouts are selected at random from the group. Find the probability distribution of number of selected scouts who are well trained in first aid. Also, find the mean of the distribution. What value is depicted by a well trained scout?

20. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = 4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, then find a vector of magnitude 6 units which is parallel to the vector $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$.

21. A pair of dice is thrown 4 times. If getting a doublet is a successes, then find the probability distribution of number of successes. [Delhi 2008]

22. For $n > 0$, find the value of

$$\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx.$$

23. Show that $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at $x = 1$.

Section D

24. A dealer deals in two items A and B. He has ₹ 15000 to invest and a space to store atmost 80 pieces. Item A costs him ₹ 300 and item B costs him ₹ 150. He can sell items A and B at profits of ₹ 40 and ₹ 25 respectively. Assuming that he can sell all that he buys, formulate the above as a linear programming problem for maximum profit and solve it graphically. [Delhi 2010C]

25. If $A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations.

$8x - 4y + 2 = 5, 10x + 6z = 4$ and $8x + y + 6z = \frac{5}{2}$

26. A wire of length 28 m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the lengths of two pieces so that the combined area of circle and square is minimum?

27. Find the coordinates of foot of perpendicular drawn from the point (0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. Also, find the length of perpendicular.

28. Find the area enclosed by the parabola $4y = 3x^2$ and the line $3x - 2y + 12 = 0$.

29. Show that the function f in $A = R - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}.

Answers with Explanation

1. $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$

$\left[\because \tan\frac{\pi}{4} = 1 \text{ and } \sin\frac{\pi}{6} = \frac{1}{2}\right]$

$= \frac{\pi}{4} - \frac{\pi}{6}$

$= \frac{\pi}{12}$

2. Given, $f(x) = \sin x$ and $g(x) = 5x^2$

$\therefore \quad gof(x) = g[f(x)] = g(\sin x)$

$= 5(\sin x)^2 = 5\sin^2 x$

3. $\because \quad \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x + 6 \\ 4x + 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

On equating the corresponding elements, we get

$x + 6 = 5$

$\Rightarrow \quad x = -1$

4. $\because \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$

$= \hat{\mathbf{i}}(-14 + 14) - \hat{\mathbf{j}}(2 - 21) + \hat{\mathbf{k}}(-2 + 21) = 19\hat{\mathbf{j}} + 19\hat{\mathbf{k}}$

$\therefore \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

5. We have $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{1+x}{1-x} - x}{1 + \left(\frac{1+x}{1-x}\right) \cdot x}\right] = \frac{\pi}{4}$

$\Rightarrow \quad \frac{1 + x^2}{1 + x^2} = 1 \qquad \left[\because \tan\frac{\pi}{4} > 1\right]$

6. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 5a+4c & 5b+4d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

On equating and solving, we get

$a = -3, b = -14, c = 4$ and $d = 17$

7. $\because f(x) = 2x - |x| = \begin{cases} 2x - x, & \text{when } x \geq 0 \\ 2x - (-x), & \text{when } x < 0 \end{cases}$

$= \begin{cases} x, & \text{when } x \geq 0 \\ 3x, & \text{when } x < 0 \end{cases}$

Now, $f(0) = 0$

$\text{RHL} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x) = 0$

and $\text{LHL} = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (3x) = 0$

$\because$ (LHL at $x = 0$) = (RHL at $x = 0$) = $f(0)$

Hence, $f(x)$ is continuous at $x = 0$. **Hence proved.**

8. Given, $f(x) = x^3 - 3x^2 + 3x - 100$

$\Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$= 3(x-1)^2 \geq 0$ for all real x

Hence, f is an increasing function in R. **Hence proved.**

9. Let $I = \int \frac{x^3}{1+x^8} = \int \frac{x^3}{1+(x^4)^2}\,dx$

Put $x^4 = \tan\theta$

$\Rightarrow 4x^3 dx = \sec^2\theta\, d\theta$

$\therefore I = \int \frac{1}{1+\tan^2\theta} \cdot \frac{\sec^2\theta}{4}\,d\theta = \frac{1}{4}\theta + C = \frac{\tan^{-1}(x^4)}{4} + C$

10. Let $g(x) = 1 - x + |x|$ and $h(x) = |x|$

Then, $(hog)(x) = h(g(x)) = |1 - x + |x|| = f(x)$

Since, $|x|$ is a continuous function and $1 - x$ is a continuous function as it is a polynomial function.

Thus, $1 - x + |x|$ is a continuous function. **Hence proved.**

11. Let the three points be A, B and C respectively.

Then, $\mathbf{OA} = \mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, \mathbf{OB} = 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$

and $\mathbf{OC} = -7\mathbf{b} + 10\mathbf{c}$

Now, $\mathbf{AB} = \mathbf{a} + 5\mathbf{b} - 7\mathbf{c}$

$\mathbf{AC} = -\mathbf{a} - 5\mathbf{b} + 7\mathbf{c} = (-1)\,\mathbf{AB}$

$\therefore AC \| AB$

Hence, points A, B and C are collinear. **Hence proved.**

12. Let A be the event of getting an odd number on first die and B be the event of getting a multiple of 3 on the second die.

Then, $A = \{1,3,5\}$ and $B = \{3,6\}$

$\therefore P(A) = \frac{1}{2}$

and $P(B) = \frac{1}{3}$

Hence, required probability

$= P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

13. Here, $A^2 = A \cdot A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$

$6A = 6\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix}$

and $17I = 17\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$

$\therefore A^2 - 6A + 17I = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now, $A^2 - 6A + 17I = O$

$\Rightarrow A^2 - 6A = -17I$

$\Rightarrow A - 6I = -17A^{-1}$

[multiplying both sides by A^{-1}]

$\therefore A^{-1} = \frac{1}{17}(6I - A) = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

14. Let $I = \int \sin x \sin 2x \sin 3x\,dx = \frac{1}{2}\int \sin x\,(2\sin 2x \cdot \sin 3x)\,dx$

$= \frac{1}{2}\int \sin x\,(\cos x - \cos 5x)\,dx$

$= \frac{1}{4}\int \sin 2x\,dx - \frac{1}{4}\int (\sin 6x - \sin 4x)\,dx$

$= -\frac{\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$

Or

Let $I = \int \frac{e^x(x^2+1)}{(x+1)^2}\,dx = \int e^x\left(1 - \frac{2x}{(x+1)^2}\right)dx$

$= e^x - 2\int e^x\left[\frac{1}{x+1} + \frac{(-1)}{(x+1)^2}\right]dx$

$= e^x - 2e^x \cdot \frac{1}{x+1} + C$

15. We have, $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{x+2}{1-2x}\right)$

$= \tan^{-1}\left(\frac{1-x}{1+x\cdot 1}\right) - \tan^{-1}\left(\frac{x+2}{1-2\cdot x}\right)$

$= \tan^{-1}1 - \tan^{-1}2 - 2\tan^{-1}x$

On differentiating both sides w.r.t. x, we get

$f'(x) = -\frac{2}{1+x^2}$

16. $\because 5x + 3 = A\frac{d}{dx}(x^2 + 4x + 10) + B$

$\Rightarrow 5x + 3 = 2Ax + (4A + B)$

On equating their coefficients, we get

$A = \frac{5}{2}$ and $B = -7$

Let $I = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2+4x+10}}\,dx$

$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx - 7\int \frac{dx}{\sqrt{x^2+4x+10}}$

$= \frac{5}{2}I_1 - 7I_2$ ...(i)

where, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}\,dx$ and $I_2 = \int \frac{dx}{\sqrt{x^2+4x+10}}$

$\therefore \quad I_1 = 2\sqrt{x^2+4x+10} + C_1$

and $\quad I_2 = \log|(x+2) + \sqrt{x^2+4x+10}| + C_2$

On putting the values of I_1 and I_2 in Eq. (i), we get

$I = 5\sqrt{x^2+4x+10} - 7\log|(x+2) + \sqrt{x^2+4x+10}| + C$

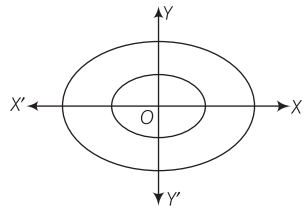
17. The family of ellipses having foci on X-axis and centre at the origin, is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

On differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad \frac{2y}{b^2}\cdot\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\Rightarrow \quad \frac{y\left(\frac{dy}{dx}\right)}{x} = \frac{-b^2}{a^2}$$



Again differentiating w.r.t. x, we get

$$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

which is the required differential equation.

18. The vector equation of required line is

$$\mathbf{r} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) + \lambda(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \qquad \ldots(i)$$

If line (i) meets the plane $\mathbf{r}\cdot(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + 2 = 0$, then

$[(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) + \lambda(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}})]\cdot(6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) + 2 = 0$

$$\Rightarrow \quad \lambda = \frac{1}{35}$$

On putting $\lambda = \frac{1}{35}$ in Eq. (i), we get

$$\mathbf{r} = \frac{1}{35}(76\hat{\mathbf{i}} - 108\hat{\mathbf{j}} - 170\hat{\mathbf{k}})$$

Hence, required point of intersection is $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-34}{7}\right)$.

19. Let X be number of selected scouts who are well trained in first aid.

Here, random variable X may have values 0, 1 and 2.

Using formula $P(X = r) = {}^nC_r p^r q^{n-r}$, the required probability distribution is given below:

X	0	1	2
$P(X)$	$\frac{38}{245}$	$\frac{120}{245}$	$\frac{87}{245}$

$\therefore \quad$ Mean $= \Sigma X_i P_i = \frac{120}{245} + \frac{174}{245} = \frac{294}{245}$

The value like cooperation is expected from a well trained scout.

20. $\because \mathbf{r} = 2\mathbf{a} - \mathbf{b} + 3\mathbf{c}$

$= 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}} - 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}} + 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$= \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Since, the required vector has magnitude 6 units and parallel to $\mathbf{r}$.

$$\therefore \quad \mathbf{r} = 6\left[\frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}}\right] = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

21. We know that the probability of getting doublet $(p) = \frac{1}{6}$

Probability of not getting doublet $(q) = \frac{5}{6}$

$\because \quad n = 4$

Let X denotes the number of doublets.

Here, random variable X can take values 0, 1, 2, 3 and 4.

Using formula $P(X = r) = {}^nC_r p^r q^{n-r}$, the required probability distribution is given below:

X	0	1	2	3	4
$P(X)$	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{75}{648}$	$\frac{5}{324}$	$\frac{1}{1296}$

22. Let $\quad I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots(i)$

Then, $\quad I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get

$$2I = 2\pi\cdot 2\cdot 2\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\therefore \quad I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots(iii)$$

Let $\quad I_1 = \int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots(iv)$

Then, $\quad I_1 = \int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x + \sin^{2n}x}\,dx \qquad \ldots(v)$

From Eqs. (iv) and (v), we get

$$2I_1 = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Now, from Eq. (iii), we get

$$I = 4\pi\cdot\frac{\pi}{4} = \pi^2$$

23. To show that $f(x)$ is continuous at $x = 1$, we have to show

$$(\text{LHL at } x = 1) = (\text{RHL at } x = 1) = f(1) \qquad \ldots(i)$$

Now, $\quad \text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 5x - 4 = 1$

and $\quad \text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x^3 - 3x) = 1$

Also, $\quad f(1) = 5(1) - 4 = 1$

$\therefore f(1)$ is continuous at $x = 1$. **Hence proved.**

Do You Know!!!

The sum of a sequence of odd numbers gives you the sequence for the perfect squares.
$1 = 1,\ 1+3 = 4,\ 1+3+5 = 9, 1+3+5+7 = 16,\ 1+3+5+7+9 = 25$ and so on.

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INSTRUCTIONS

- Question Numbers 1-4 in **Section A** are Very Short Answers carrying 1 mark each.
- Question Numbers 5-12 in **Section B** are Short Answers carrying 2 marks each.
- Question Numbers 13-23 in **Section C** are Long Answers I carrying 4 marks each.
- Question Numbers 24-29 in **Section D** are Long Answers II carrying 6 marks each.

Section A

1. If $f : R \to R$ is defined by $f(x) = 3x + 2$, then find $f[f(x)]$. [Foreign 2011; Delhi 2010]

2. Write the principal value of

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right).$$

[All India 2011, 2009C]

3. Find the simplified value of

$$\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} + \sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}.$$

[Delhi 2012]

4. Find a vector in the direction of $\mathbf{a} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ which has magnitude 6 units.

Section B

5. Find the value of

$$\tan\left[\frac{1}{2}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}\right];\ |x| < 1,\ y > 0$$

and $xy < 1$.

6. Find the equation of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$.

7. If $A = \begin{bmatrix}2 & 3 & 4\\ 1 & 2 & 3\\ 4 & -5 & 7\end{bmatrix}$ and $B = \begin{bmatrix}2 & 5 & -1\\ 3 & -1 & 2\\ 7 & 2 & 3\end{bmatrix}$, then verify that $(A + B)' = A' + B'$.

8. Evaluate $\int e^x(\sin x + \cos x)\,dx$.

9. If $P(-1, 2, 5)$ and $Q(3, 0, -2)$, then express $\mathbf{PQ}$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and find its magnitude. Also, find the unit vector in the direction of $\mathbf{PQ}$.

10. A bag contains 3 red and 5 black balls and another bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that both are (i) red (ii) black.

11. Show that the function $f(x) = \begin{cases}x^3 + 3, & \text{if } x \neq 0\\ 1, & \text{if } x = 0\end{cases}$ is discontinuous at $x = 0$.

12. If $f(x) = \begin{cases}\dfrac{x-2}{x^2 - 3x + 2}, & \text{when } x \neq 2\\ 1, & \text{when } x = 2\end{cases}$, then find $f'(2)$.

Section C

13. If $A = \begin{bmatrix}2 & 1 & 3\\ 4 & 1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}1 & -1\\ 0 & 2\\ 5 & 5\end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

14. Evaluate $\int \frac{\log(\sin x)}{\tan x}\,dx$.

Or

Evaluate $\int \frac{3x + 5}{\sqrt{x^2 - 8x + 7}}\,dx$.

15. Find the values of a and b such that the function defined as

$$f(x) = \begin{cases}x + 2, & \text{if } x \leq 2\\ ax + b, & \text{if } 2 < x < 5\\ 3x - 2, & \text{if } x \geq 5\end{cases}$$

is continuous. [Delhi 2010]

16. Evaluate $\int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$. [Delhi 2010C]

17. For what value of k is the function

$$f(x) = \begin{cases}2x + 1, & \text{if } x < 2\\ a, & \text{if } x = 2\\ x + 1, & \text{if } x > 2\end{cases}$$

continuous at $x = 2$?

Find the value of a. Also, discuss the continuity of $f(x)$ at $x = 3$. [Delhi 2009C]

18. Solve the differential equation $xdy + (y - x^3)\,dx = 0$. [All India 2011]

19. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx$.

20. If the position vectors of the points, P, Q, R and S are respectively $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$, then prove that the lines PQ and RS are parallel and the ratio of their lengths is $\frac{1}{2}$.

Or

Find the lengths of the sides of the triangle whose vertices are $A(2, 4, -1)$, $B(4, 5, 1)$, $C(3, 6, -3)$ and show that the triangle is right angled.

21. Two persons A and B throws a coin alternately till one of them gets 'head' and win the game. Find their respective probabilities of winning.

Or

In a class, 30% students fail in English, 20% students fail in Hindi and 10% students fail in English and Hindi both. A student is chosen at random. What is the probability that he will fail in English, if he has failed in Hindi?

22. Find the equation of the plane through the point $(1, 4, -2)$ and parallel to the plane $-2x + y - 3z = 7$.

23. Often it is taken that a truthful person commands more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is actually a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society?

Section D

24. Two tailors A and B earns ₹ 150 and ₹ 200 per day respectively. A can stitch 6 shirts and 4 pants per day while B can stitch 10 shirts and 4 pants per day. How many days shall each work, if it is desired to produce atleast 60 shirts and 32 pants at a minimum labour cost? Make it as an LPP and solve the problem graphically.

25. Show that the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b-c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar.

Or

Find the equation of the line of intersection of the planes $x - 2y + z = 1$ and $x + 2y - 2z = 5$ in symmetric form.

26. Using integration, find the area of triangular region whose vertices are (1, 0), (2, 2) and (3, 1). [All India 2008]

Or

Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. [All India 2008]

27. Find the equations of tangent and normal to the curves $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

28. If $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then find A^{-1} and hence solve the following system of equations.

$x - 2y + z = 0, -y + z = -2, 2x - 3z = 10$ [All India 2011C]

29. Let R be a relation defined on the set of natural number N as $R = \{(x, y) : 2x + y = 41; x \in N, y \in N\}$. Find the domain and range of the relation R. Also, check whether R is reflexive, symmetric and transitive.

Answers

1. $9x + 8$ **2.** π

3. I_2 **4.** $4\hat{i} - 2\hat{j} + 4\hat{k}$

5. $\frac{x+y}{1-xy}$ **6.** $x + 3y = 0$

8. $e^x \sin x + C$

9. $\mathbf{PQ} = 4\hat{i} - 2\hat{j} - 7\hat{k}$, $|\mathbf{PQ}| = \sqrt{69}$, $\mathbf{PQ} = \frac{4}{\sqrt{69}}\hat{i} - \frac{2}{\sqrt{69}}\hat{j} - \frac{7}{\sqrt{69}}\hat{k}$

10. (i) $\frac{9}{40}$ (ii) $\frac{1}{4}$ **12.** -1

14. $\frac{(\log \sin x)^2}{2} + C$

Or $3\sqrt{x^2 - 8x + 7} + 17 \log|(x-4) + \sqrt{(x-4)^2 - 9}| + C$

15. $a = 3, b = -2$

16. $x \log(\log x) - \frac{x}{\log x} + C$

17. $a = 3$; Continuous

18. $y = \frac{x^3}{4} + \frac{C}{x}$

19. $\pi\left(\frac{\pi}{2} - 1\right)$ **20.** *Or* $3, 3\sqrt{2}, 3$

21. $\frac{2}{3}, \frac{1}{3}$ *Or* $\frac{1}{2}$ **22.** $2x - y + 3z + 8 = 0$

23. $\frac{4}{9}$; A truthful person trusted by society, so he receive respect also.

24. Tailor $A = 5$ days, Tailor $B = 3$ days

25. *Or* $\frac{x-3}{2} = \frac{y-1}{3} = \frac{z}{4}$

26. $\frac{3}{2}$ sq units *Or* $\frac{16a^2}{3}$ sq units

27. $y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = (\sqrt{2} - 1)\left(x - 1 + \frac{1}{\sqrt{2}}\right)$, $\left(y - \frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}-1}\left(x - 1 + \frac{1}{\sqrt{2}}\right)$

28. $A^{-1} = \begin{bmatrix} 3 & -6 & -1 \\ 2 & -5 & -1 \\ 2 & -4 & -1 \end{bmatrix}$; $x = 2, y = 0$ and $z = -2$

29. $D_k = \{1, 2, 3, 4, \ldots, 20\}$, $R_k = \{1, 3, 5, 7, 9, \ldots, 39\}$
R is neither reflexive nor symmetric nor transitive.

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FUN ARCADE

Rene Descartes

Rene Descartes was a French mathematician, philosopher and writer that spent nearly all of his adult life living in the Dutch Republic. He is viewed as the inventor of modern day philosophy and his meditations on first philosophy is still required text for many philosophy departments. He is very well known for his philosophical declaration, "I think, therefore I am."

Descartes was born in La Haye, France, in March of 1596. He was educated at a boarding Jesuit school when he was eight years old. There, he studies Mathematics, music, astronomy, metaphysics, natural philosophy and ethics. Later, he earned a law degree when he was 22 from the university of Poitiers.

Descartes composed the rules for the direction of the mind, his first composition on method. In 1623, he went to La Haye to sell his property and invested the proceed in bonds. Descartes continued to publish works on philosophy and Mathematics in the later years of his life. He published the discourse on the method in 1637. In 1641, Descartes published his metaphysics work, meditations on first philosophy. In 1644, the principles of philosophy was published and this was a combination of his discourse and meditations works.

In 1647, the king of France awarded Descartes a pension. In February of 1650, he moved to Stockholm, Sweden, to be Queen Christina's philosophy tutor. There, Descartes died from pneumonia when he was 53 years old.

Descartes is regarded as the first philosopher to use reason to explore the natural science. Since, he thought that all truths were linked, Descartes searched for the explanation of the natural world by using a rational approach, by means of Mathematics and Science. This was similar to what Sir Francis Bacon had believed. His methodology of incorporating logic and Mathematics with philosophy to elucidate the physical world became metaphysical when it dealt with theology questions. This brought about his contemplation of the nature existence, as well as the mind body duality, pinpointing the soul's contact with the body at the pineal gland. This idea also contributed to the concept of dualism which was matter meeting non-matter.

Descartes developed Cartesian (analytical) geometry, which is the use of algebra to examine geometric properties. He created and empirical comprehension of rainbows, along with proposing a naturalistic account for the solar system's formation. This lead Pope Alexander VII to add his works to the list of prohibited books.

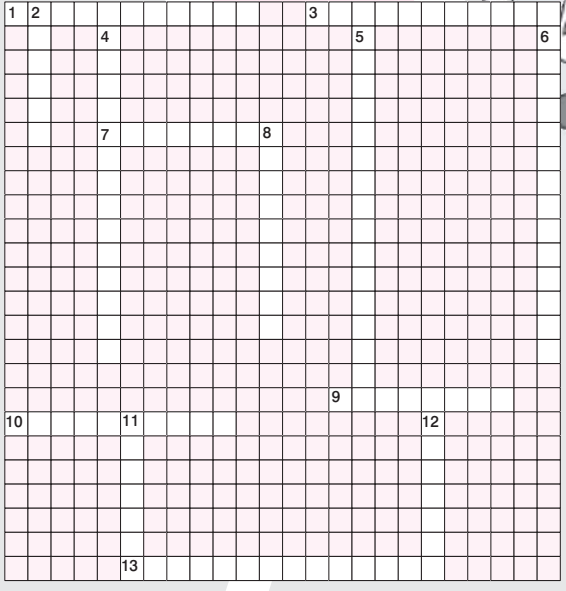
CROSSWORD

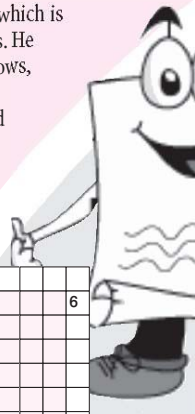
ACROSS

1. The method of removing variables from the given system of equation.
3. The term in a polynomial which contains the highest power of the variable.
7. A parallelogram that has adjacent sides unequal.
9. A number placed in a superscript position to the right of another number to indicate repeated multiplication.
10. A structure used to find the prime factorisation of positive integer.
13. The study of surfaces and solid in space including cone, cylinder, pyramids, polyhedras and spheres.

DOWN

2. An object having two dimensional shape and density but no thickness.
4. The number of rows and columns in a matrix.
5. A square matrix which is equal to the inverse of its transpose.
7. A single point at which two curves touch.
8. A curve defining the generators of a ruled surface.
11. A parallelogram that has all its side equal.
12. The curve traced out by a point on a circle rolling along a straight line.





Puzzles to Puzzle You

1. A cylindrical container has a radius of eight inches with a height of three inches. Compute how many inches should be added to either the radius or height to give the same increase in volume?

2. Here is an epitaph of the celebrated Greek mathematician of 250 AD Diophantus. Can you calculate his age from this?
Diophantus passed one-sixth of his life in childhood, one-twelfth in youth and one-seventh more as a bachelor; five years after his marriage, a son was born who died four years before his father at half his final age. How old is Diophantus?

3. A friend of mine in London has a very nice cellar. He has two large barrels in the cellar. The larger barrel is mostly empty. But the smaller barrel is only 5/6th full of wine while it can hold 536 L.
Supposing he empties the smaller barrel and fills the bigger barrel to find that the wine fills only 4/9 th of it, how much wine would the larger barrel hold when full?

4. A plane has a span of 12 m. It was photographed as it was flying directly overhead with a camera with a depth of 12 cm.
In this photo, the span of the plane was 8 mm. Can you tell how high was the plane when it was snapped?

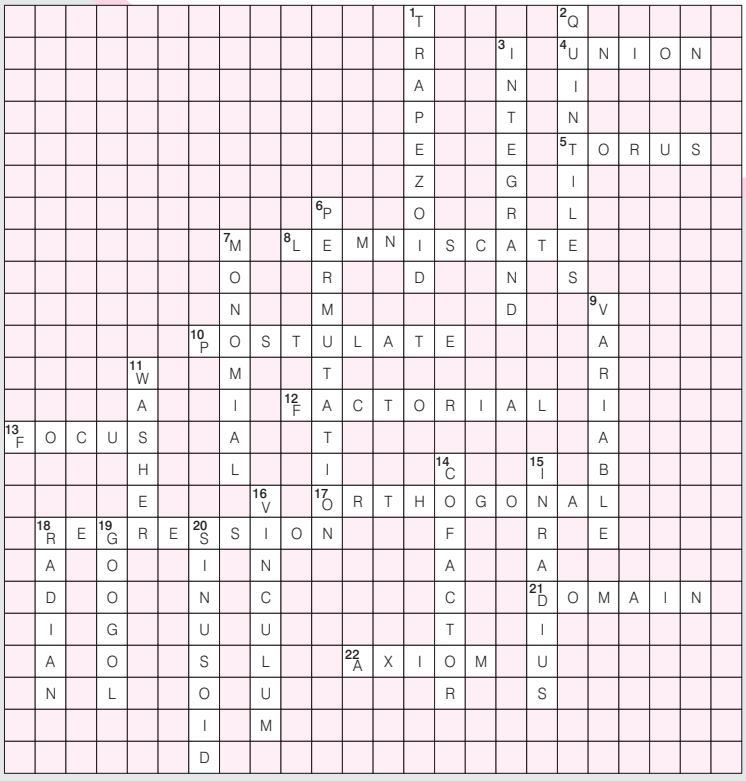
5. Sound is said to travel in air at about 1100 feet per second. A man hears the axe strike the tree 11/5 seconds after he sees it strike the tree. How far would you say the man is from the wood chopper?

6. A three digit number consists of 9, 5 and one more number. When these digits are reversed and then subtracted from the original number, the answer yielded will be consisting of the same digits arranged yet in a different order. What is the other digit?

SUDOKU

	4					3		
		5	9				2	
							4	9
	8				3			
6		7					3	2
2		1		8				
		2	7			6		5
7					6			
			3					4

January 2017 issue Answer



Puzzles to Puzzle you Answers January Issue

1. 630 **2.** 19 years
3. 1/2 **4.** 104%
5. Daughter-4, Son-3
6. 13 inches

SUDOKU

5	2	6	1	9	7	4	8	3
1	8	3	6	2	4	5	7	9
9	4	7	5	3	8	2	1	6
8	7	9	2	5	6	1	3	4
2	3	4	7	1	9	6	5	8
6	1	5	8	4	3	7	9	2
3	9	2	4	7	5	8	6	1
4	5	8	3	6	1	9	2	7
7	6	1	9	8	2	3	4	5

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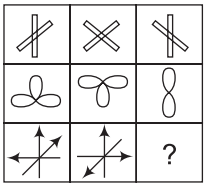
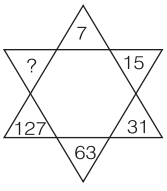
1. If ₹ 10 is allowed as true discount on a bill of ₹ 110 due at the end of a certain time, then what will be the discount allowed on the same sum due at the end of double the time?
2. Look at this series 8, 43, 11, 41, ?, 39, 17, . . .
   What number should come in place of (?) question mark?
3. Fill the suitable figure that would complete the figure matrix.



4. If 'NEW YORK' can be encrypted as 'PGYAQTM', how can you code the word 'CHARLOTTE'?
5. Find the missing number.



6. Atul is twice as old as Rhythm when Atul was as old as Rhythm is now. The combined age of Atul and Rhythm is 112 yr. How old are Atul and Rhythm now?
7. What letter comes two to the right of the letter which is immediately to the left of the letter that comes three to the right of the letter that comes midway between the letter two to the left of the letter *C* and the letter immediately to the right of the letter *F*?
8. You have 9 cubic blocks. What is the minimum number that needs to be taken away in order to construct a solid cube with none left over?
9. A bag contains 64 balls of eight different colours. There are eight of each colour (including red). What is the least number you would have to pick, without looking, to be sure of selecting 3 red balls?
10. In a group of 28 students, 7 take French, 10 take Spanish and 4 take both languages. The student taking both French and Spanish are not counted with 7 taking French or 10 taking Spanish. How many students are not taking either French or Spanish?

KNOWLEDGE Coefficient Quizzer (No. 27)

Write your answers on separate sheets and send to us along with this coupon

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